More counterexamples to Happel's question and Snashall-Solberg's conjecture

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Abstract In this paper we provide more counterexamples to Happel's question and Snashall-Solberg's conjecture which generalize many counterexamples to these conjectures studied in the literature. In particular, we show that a family of $\mathbb{Z}_n \times \mathbb{Z}_n$-Galois covering algebras of quantized exterior algebra A_q in two variables answer negatively to Happel's question, and meanwhile, the one-point coextensions of $\mathbb{Z}_n$ and $\mathbb{Z}_n \times \mathbb{Z}_m$-Galois covering algebras of A_q negate the Snashall-Solberg's conjecture.

Keywords Hochschild cohomology ring; Happel's question; Snashall-Solberg's conjecture; Koszul dual; graded center

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1. Introduction

Let Λ be a finite-dimensional k-algebra (associative with identity) over a field k. Denote by Λ^e the enveloping algebra of Λ, i.e., the tensor product $\Lambda \otimes_k \Lambda^{op}$ of the algebra Λ and its opposite Λ^{op}. Then by Cartan-Eilenberg [**1**] the i-th Hochschild homology and cohomology groups of Λ are identified with the k-spaces

$$\mathrm{HH}_i(\Lambda) = \mathrm{Tor}_i^{\Lambda^e}(\Lambda, \Lambda), \quad \mathrm{HH}^i(\Lambda) = \mathrm{Ext}^i_{\Lambda^e}(\Lambda, \Lambda)$$

respectively. The Hochschild cohomology ring $\mathrm{HH}^*(\Lambda) = \bigoplus_{i=0}^{\infty} \mathrm{HH}^i(\Lambda)$ has a so-called Gerstenhaber algebra structure under the cup product and the Gerstenhaber bracket [**2**]. It is well known, as a noncommutative analogy of differential forms and polyvector fields, that Hochshild homology and cohomology of an associative (noncommutative) algebra have been a starting point of noncommutative geometry and play an important role due to the classic Hochschild-Kostant-Rosenberg theorem.

It is also well known that the homological properties of an algebra are closely related to the properties of its Hochschild (co)homology groups. For example, if a finite dimensional algebra over an algebraically closed field has finite global dimension, then all its higher Hochschild cohomology groups vanish. The inverse is known as Happel's question and it has been shown that the conjecture does not hold for the quantized exterior algebra $A_q = k\langle x, y\rangle/(x^2, xy + qyx, y^2)$ (or more generally, for the quantized complete intersection) when $q \in k^* = k \setminus \{0\}$ is not a root of unity in [**3, 4, 5, 6**]. However, the homology version of Happel's question comes to be known as "Han's conjecture" and remains still open [**7**].

Motivated by support variety for finitely generated modules over group algebras defined by Carlson in [**8**], Snashall and Solberg developed support variety theory of finitely generated modules over a finite-dimensional algebra in [**9**]. They found that the finiteness condition of Hochschild cohomology ring modulo nilpotence $\mathrm{HH}^*(\Lambda)/\mathcal{N}$ played an important role in support variety theory, where $\mathcal{N}$ denotes the ideal of $\mathrm{HH}^*(\Lambda)$ generated by all homogeneous nilpotent elements. Moreover, they also conjectured that the Hochschild cohomology ring modulo nilpotence of any finite-dimensional algebra is a finitely generated algebra, and the conjecture was proved to be true for many classes of algebras, such as finite-dimensional algebras of finite global dimension[**3**], finite-dimensional monomial algebras[**10, 11**], finite-dimensional self-injective algebras of finite representation type over an algebraically closed field[**12**], any block of a group ring of a finite group[**13, 14**] and so on. Until 2008, Xu F. provided the first counterexample to the conjecture by studying the Hochschild cohomology ring modulo nilpotence of a seven-dimensional category algebra in the case of char$k = 2$ [**15**], which is isomorphic to a Koszul algebra [**16**]. Furthermore,

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it was proved that the Hochschild cohomology ring modulo nilpotence of the above Koszul algebra as well as its quantized algebra is not a finitely generated algebra irrespective of the characteristic of the base field k[**16, 17**].

Let Λ_q be the algebra introduced in the first paragraph of the section 2, which arises from a formal deformation with infinitesimal in $\mathrm{HH}^2(\Lambda)$ and occurs in the study of the Drinfeld double of the generalized Taft algebras and of the representation theory of $U_q(\mathsf{sl}_2)$. Snashall and Taillefer proved the Hochschild cohomology rings modulo nilpotence of Λ and Λ_q are finitely generated commutative algebras of Krull dimension two and hence Snashall-Solberg's conjecture holds for this class of algebras[**18, 19**]. However, Parker and Snashall showed in [**20**] that Λ_q is an infinite family of counterexamples to Happel's question when $\zeta = q_0q_1\cdots q_{m-1}$ is not a root of unity. Furthermore, we prove that, for the algebra Γ_q introduced in the first paragraph of the section 2 which can be viewed as a one-point coextension of Λ_q, $\mathrm{HH}^*(\Gamma_q)/\mathcal{N}$ is not a finitely generated algebra if $\zeta = q_0q_1\cdots q_{m-1}$ is a root of unity and thus provides an infinite family of counterexamples to Snashall-Solberg's conjecture.

Note that, when $q_0 = q_1 = \cdots = q_{m-1}$, the algebra Λ_q is just a $\mathbb{Z}_n$-Galois covering algebra of the quantized exterior algebra A_q [**21**], while the algebra Γ_q can be viewed as a one-point coextension of the $\mathbb{Z}_n$-covering algebra Λ_q. So it seems that the following question arises naturally: *if an algebra A (for example, the quantized exterior algebra A_q) answers negatively to Happel's question, does it so for any finite Galois covering algebra $\widetilde{A}$ of A, and meanwhile, will the one-point (co)extension of $\widetilde{A}$ provide a family of counterexamples to the Snashall-Solberg's conjecture?* Let G be a finite group, A a G-graded k-algebra, and $\widetilde{A}$ the covering algebra with the Galois group G. It was shown in [**22, 23**] that there is a ring monomorphism from $\mathrm{HH}^i(\widetilde{A})$ to $\mathrm{HH}^i(A)$ for $i \geq 0$. As a consequence, if A is a counterexample to Happel's question, then so is $\widetilde{A}$. Indeed, this is the case for the $\mathbb{Z}_2$-graded quantized exterior algebra A_q and its Galois covering algebra with Galois group $\mathbb{Z}_2$ (even more generally, $\mathbb{Z}_n$) [**24, 20**]. However, if A is only a k-algebra (unnecessarily G-graded), then there is only a monomorphism from $\mathrm{HH}^i(\widetilde{A})^G$ to $\mathrm{HH}^i(A)$ for $i \geq 0$, and the explicit descriptions of these maps for $i = 0, 1$ are provided in [**25**].

In this paper, we first employ Snashall and Taillefer's strategy in [**19**] to consider the structure of the Hochschild cohomology rings modulo nilpotence $\mathrm{HH}^*(\Lambda_q)/\mathcal{N}$ of the algebras Λ_q by computing the graded center of its Koszul dual $E(\Lambda_q)$ in the section 2. As a consequence, we show that they are not finitely generated as algebras when ζ is a root of unity, and thus provide more counterexamples to Snashall-Solberg's conjecture, which include and generalize all the counterexamples studied in [**15, 16, 17**]. Next, we consider a family of algebras $\Lambda_q^{m,n}$ as well as their one-point coextensions $\Gamma_q^{m,n}$, where $q = (q_{00}, q_{01}, \cdots, q_{n-1,m-1}) \in (k^*)^{mn}$. In the case that $q_{ij} = q_{00}$ for all i, j, $\Lambda_q^{m,n}$ is just a covering algebra of the quantized exterior algebra A_q with the Galois group $\mathbb{Z}_n \times \mathbb{Z}_m$. We determine the structure of Hochschild cohomology ring of $\Lambda_q^{n,n}$ and show that $\Lambda_q^{n,n}$ answers negatively to Happel's question when $\xi = \prod_{i,j=0}^{n-1} q_{ij} \in k^*$ is not a root of unity in the section 3, and meanwhile, $\Gamma_q^{m,n}$ provides an infinity family of counterexamples to Snashall-Solberg's conjecture when $\eta = \prod_{i=0}^{n-1}\prod_{j=0}^{m-1} q_{ij} \in k^*$ is a root of unit in the section 4 as expected.

2. Graded center of $E(\Gamma_q)$

Throughout this section we always assume that $\Lambda = kQ/I$ is a class of selfinjective Koszul algebras, where the quiver Q is of the form in the left hand side of Figure 1 below, and the ideal I is generated by the set $R = \{a_ia_{i+1}, b_{i-1}b_{i-2}, a_ib_i - b_{i-1}a_{i-1} \mid 0 \leq i \leq m-1\}$. Let Λ_q, $q = (q_0, q_1, \cdots, q_{m-1}) \in (k^*)^m$, be their socle deformations (i.e. Λ_q are selfinjective with $\Lambda_q/\mathrm{soc}(\Lambda_q) \cong \Lambda/\mathrm{soc}(\Lambda)$), see also [**18, 19**]. Throughout we always assume that all the subscripts of arrows are identified with their residues modulo m.

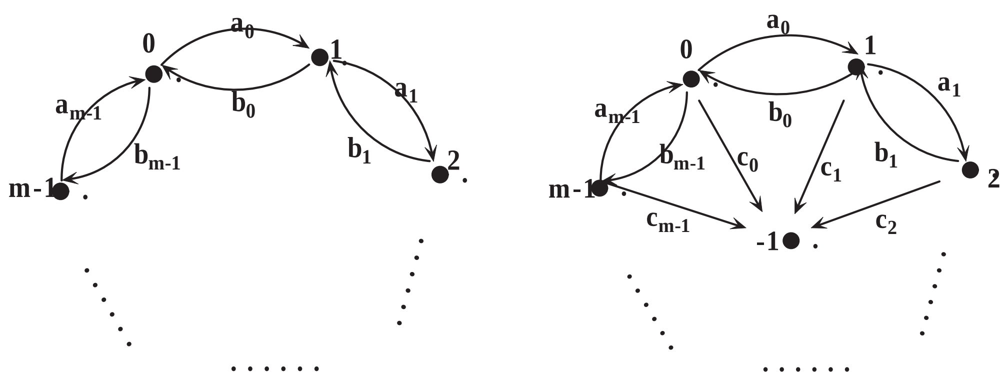

Figure 1. The quivers Q and Q'

Let $\Gamma_q = kQ'/I'_q$, where Q' is the finite quiver with $m+1$ vertices $\{0, 1, \ldots, m-1\} \cup \{-1\}$ and $3m$ arrows pictured in Figure 1 as well, and I'_q is the ideal generated by $R' = \{a_i a_{i+1}, b_{i+1} b_i, q_i a_i b_i - b_{i-1} a_{i-1}, a_i c_{i+1} \mid 0 \leq i \leq m-1, a_m = a_0\}$ and $q = (q_0, q_1, \cdots, q_{m-1}) \in (k^*)^m$. In the case of $m = 1$, Γ_q is isomorphic to the quantized Koszul algebra studied in [**17**] (in which the "commutative" relation is $ab + qba$) and used to provide a family of counterexamples to Snashall-Solberg's conjecture. Throughout the section we assume $m \geq 2$.

Denote by e_i the trivial path in kQ' and write the composition of arrows from left to right. Note that the left length lexicographic order by choosing $e_0 < \cdots < e_{m-1} < e_{-1} < a_0 < \cdots < a_{m-1} < b_0 < \cdots < b_{m-1} < c_0 < \cdots < c_{m-1}$ provides an admissible order on kQ' and thus the set R' is just a (noncommutative) quadratic reduced Gröbner basis of I'_q[**26**]. So Γ_q is Koszul by [**27**].

Recall that the Ext-algebra $E(\Gamma_q)$ is just the Koszul dual of Γ_q. Thus $E(\Gamma_q) = kQ^{op}/I'^{\perp}_q$, where $I'^{\perp}_q = \langle q_i^{-1}(a_i b_i)^o + (b_{i-1}a_{i-1})^o, (b_i c_i)^o \rangle$ and $x^o \in Q^o$ denotes the opposite arrow of the arrow x in Q. Moreover, any left kQ^{op}-module can be viewed as a right kQ-module, so we may consider $E(\Gamma_q)$ as the quotient of kQ modulo the ideal generated by $q_i^{-1} a_i b_i + b_{i-1} a_{i-1}, b_i c_i$ for $i = 0, 1, \cdots, m-1$.

In a similar way to [**19**], we denote by γ_i^s and δ_i^t the paths $a_i a_{i+1} \cdots a_{i+s-1}$ and $b_{i+t-1} \cdots b_{i+1} b_i$ respectively. Unless otherwise specified, we do not distinguish a path with its image in $E(\Gamma_q)$. Thus any typical monomial in $E(\Gamma_q)$ has the form $\gamma_i^s \delta_j^t$ for some integers s, t and $0 \leq i, j \leq m-1$. The algebra $E(\Gamma_q)$ is a bigrading algebra graded with the lengths of paths and with the degree induced by choosing the degree of e_i, a_j, c_j and b_j to be $0, 1, 1, -1$ respectively. Thus any monomial element $\gamma_i^s \delta_j^t$ has the length $s + t$ and degree $s - t$. we denote by $|z|$ the length of a length-homogeneous element z in $E(\Gamma_q)$. Denote by $Z_{gr}(E(\Gamma_q))$ the graded center of $E(\Gamma_q)$.

It is easy to see that $z \in Z_{gr}(E(\Gamma_q))$ if and only if z satisfies the following conditions:

(1) $e_j z = z e_j$, for $j = -1, 0, 1, \cdots, m-1$;

(2) $a_j z = (-1)^{|z|} z a_j$, for $0 \leq j \leq m-1$;

(3) $b_j z = (-1)^{|z|} z b_j$, for any $0 \leq j \leq m-1$;

(4) $c_j z = (-1)^{|z|} z c_j$, for any $0 \leq j \leq m-1$.

Lemma 2.1. If a homogeneous element $z \in Z_{gr}(E(\Gamma_q))$, then $z \in k$, or z has the form

$$z = \sum_{i=0}^{m-1} u_i \gamma_i^{s_0} \delta_i^{t_0}, \quad u_i \in k$$

with $s_0 \equiv t_0 (\mathrm{mod}\ m)$, $t_0 \geq 1$, and for $0 \leq j \leq m-1$,

$$u_{j+1} = (-1)^{s_0} (q_{j+1} \cdots q_{j+t_0})^{-1} u_j = (-1)^{t_0} (q_{j+1} \cdots q_{j+s_0})^{-1} u_j. \tag{2-1}$$

Proof. By (1), we can write $z = \sum_{i=-1}^{m-1} e_i z e_i$. Note that for any $0 \le i \le m-1$, a typical monomial in $e_i E(\Gamma_q) e_i$ has the form $\gamma_i^s \delta_i^t$, where $s, t \ge 0$, and $s \equiv t (\mathrm{mod}\, m)$. In particular, $e_{-1} E(\Gamma_q) e_{-1} = e_{-1}$. Moreover, $Z_{gr}(E(\Gamma_q))$ is generated by the elements which are both length homogeneous and degree homogeneous. Therefore, if the length of z is 0, then $z = \sum_{i=-1}^{m-1} d_i e_i$, where $d_i \in k$; otherwise, z has the form $\sum_{i=0}^{m-1} u_i \gamma_i^{s_i} \delta_i^{t_i}$, where $u_i \in k$, $s_i, t_i \ge 0$, $s_i \equiv t_i (\mathrm{mod}\, m)$. The degree homogeneity implies that $s_i - t_i = s_0 - t_0$ and the length homogeneity implies that $s_i + t_i = s_0 + t_0 > 0$, and hence we have $s_i = s_0$ and $t_i = t_0$ for $i = 0, 1, \cdots, m-1$. So

$$z = \sum_{i=0}^{m-1} u_i \gamma_i^{s_0} \delta_i^{t_0}.$$

Here we also take the subscripts modulo m (in particular, $u_0 = u_m$).

We next consider the condition (2). If the length of $z = \sum_{i=-1}^{m-1} d_i e_i$ is zero, then $d_{j+1} a_j = a_j \sum_{i=-1}^{m-1} d_i e_i = (\sum_{i=-1}^{m-1} d_i e_i) a_j = d_j a_j$, and we have $z = d_0 \sum_{i=0}^{m-1} e_i + d_{-1} e_{-1}$. On the other hand, if the length of z is not zero, we have

$$a_j z = u_{j+1} a_j \gamma_{j+1}^{s_0} \delta_{j+1}^{t_0} = u_{j+1} \gamma_j^{s_0+1} \delta_{j+1}^{t_0}$$

and

$$z a_j = u_j \gamma_j^{s_0} \delta_j^{t_0} a_j = (-1)^{t_0} u_j (q_{j+1} \cdots q_{j+t_0})^{-1} \gamma_j^{s_0+1} \delta_{j+1}^{t_0}.$$

The condition (2) implies that $u_{j+1} = (-1)^{s_0} (q_{j+1} \cdots q_{j+t_0})^{-1} u_j \neq 0$ and similarly, the condition (3) implies that $u_{j+1} = (-1)^{t_0} (q_{j+1} \cdots q_{j+s_0})^{-1} u_j \neq 0$ for $0 \le j \le m-1$.

By the condition (4), we know that if the length of z is zero, then $d_{-1} c_j = c_j (d_0 \sum_{i=0}^{m-1} e_i + d_{-1} e_{-1}) = (d_0 \sum_{i=0}^{m-1} e_i + d_{-1} e_{-1}) c_j = d_0 c_j$, so $d_{-1} = d_0$, and thus $z = d_0 (\sum_{i=-1}^{m-1} e_i) = d_0$. Otherwise, $z = \sum_{i=0}^{m-1} u_i \gamma_i^{s_0} \delta_i^{t_0}$ satisfies $0 = c_j z = (-1)^{s_0+t_0} z c_j = (-1)^{s_0+t_0} \sum_{i=0}^{m-1} u_i \gamma_i^{s_0} \delta_i^{t_0} c_j$ for all $0 \le j \le m-1$ in $E(\Gamma_q)$, which forces that $t_0 \ge 1$ by the definition of ${I'}_q^{\perp}$. We complete the proof of the lemma. $\square$

Remark. Comparing with the result in [**19**], we have $Z_{gr}(E(\Gamma_q)) \setminus k = \{z = \sum_{i=0}^{m-1} u_i \gamma_i^{s_0} \delta_i^{t_0} \in Z_{gr}(E(\Lambda_q)) \mid t_0 \ge 1\}$. Using the formula (2-1) recursively, one can obtain that

$$u_i = (-1)^{i s_0} \prod_{k=1}^{i} (q_k \cdots q_{k+t_0-1})^{-1} u_0 = (-1)^{i t_0} \prod_{k=1}^{i} (q_k \cdots q_{k+s_0-1})^{-1} u_0,$$

for $i = 1, 2, \ldots, m-1$. In particular,

$$u_0 = u_m = (-1)^{m s_0} (q_0 \cdots q_{t_0-1})^{-1} (q_1 \cdots q_{t_0})^{-1} \cdots (q_{m-1} \cdots q_{m-2+t_0})^{-1} u_0.$$

Since $u_0 \neq 0$, we have $(q_0 \cdots q_{t_0-1})^{-1} (q_1 \cdots q_{t_0})^{-1} \cdots (q_{m-1} \cdots q_{m-2+t_0})^{-1} = (-1)^{m s_0}$. Let $\zeta = q_0 q_1 \cdots q_{m-1}$, then $\zeta^{t_0} = (-1)^{m s_0}$. Similarly, $\zeta^{s_0} = (-1)^{m t_0}$.

Proposition 2.2. If ζ is not a root of unity, then $Z_{gr}(E(\Gamma_q)) = k$.

Proof. For any element $z \in Z_{gr}(E(\Gamma_q))$, if the length of z is not zero, then $\zeta^{t_0} = (-1)^{m s_0}$ and $\zeta^{s_0} = (-1)^{m t_0}$. Since ζ is not a root of unity, then $s_0 = t_0 = 0$, this yields a contradiction. Thus the length of z is zero. By Lemma 2.1 we have $z \in k$. On the other hand, it is evident that $k \subseteq Z_{gr}(E(\Gamma_q))$, therefore, $Z_{gr}(E(\Gamma_q)) = k$ as desired. $\square$

Now we assume that ζ is a primitive d-th root of unity, that is, $d \ge 1$ is the minimal integer such that $\zeta^d = 1$. The proof of the following proposition is similar to that of [**19**, Prop.2.4, 2.5] and hence we omit all the details and leave only the sketch of the proof.

Proposition 2.3. Suppose that ζ is a primitive d-th root of unity. Then

$$Z_{gr}(E(\Gamma_q)) \cong \begin{cases} (k[x,y,w]/\langle w^{2m} - \epsilon_d xy \rangle)_{x^*}, & \text{if } m \text{ is odd, char} k \neq 2, \text{ and } d \equiv 2 (\mathrm{mod}\ 4); \\ (k[x,y,w]/\langle w^{m} - \epsilon_d xy \rangle)_{x^*}, & \text{otherwise.} \end{cases}$$

where $(k[x,y,w]/\langle w^p-\epsilon_d xy\rangle)_{x^*}$ denotes the subalgebra of $k[x,y,w]/\langle w^p-\epsilon_d xy\rangle$ that does not contain the subspace spanned by the x^i for $i=1,2,\cdots$, and

$$\epsilon_d=\begin{cases}\prod_{l=1}^{m-1}\prod_{k=1}^{ld}(-1)^{md/2}(q_k\cdots q_{k+d-1})^{-1}, & \text{if } m \text{ is even or char}k=2;\\ \prod_{l=1}^{m-1}\prod_{k=1}^{ld}(q_k\cdots q_{k+d-1})^{-1}, & \text{if } m \text{ is odd, char}k\neq 2, \text{ and } \ d\equiv 0(\text{mod } 4);\\ \prod_{l=1}^{2m-1}\prod_{k=1}^{ld/2}(q_k\cdots q_{k+d/2-1})^{-1}, & \text{if } m \text{ is odd, char}k\neq 2, \text{ and } d\equiv 2(\text{mod } 4);\\ \prod_{l=1}^{m-1}\prod_{k=1}^{2ld}(q_k\cdots q_{k+2d-1})^{-1}, & \text{if } m \text{ is odd, char}k\neq 2, \text{ and } d \text{ is odd.}\end{cases}$$

Proof.* Clearly, $k\subseteq Z_{gr}(E(\Gamma_q))$. Note that $Z_{gr}(E(\Gamma_q))\setminus k=\{z=\sum_{i=0}^{m-1}u_i\gamma_i^{s_0}\delta_i^{t_0}\in Z_{gr}(E(\Lambda_q))\mid u_i\in k, t_0\geq 1\}$ by the remark above.

Case 1. m is even or char$k=2$. With a similar but lengthy analysis as in [**19**, Prop.2.4], any homogeneous element $z\in Z_{gr}(E(\Gamma_q))\setminus k$ can be written as

$$z=u_0\Big(\sum_{i=0}^{m-1}(-1)^{is}\prod_{k=1}^{i}(q_k\cdots q_{k+s-1})^{-1}\gamma_i^s\delta_i^s\Big)\Big(\sum_{i=0}^{m-1}\gamma_i^{dm}\Big)^{\alpha}\Big(\sum_{i=0}^{m-1}\delta_i^{dm}\Big)^{\alpha+h}, \tag{2-2}$$

where $s=ld$ for $0\leq l\leq m-1$. Set $w=\sum_{i=0}^{m-1}(-1)^{id}\prod_{k=1}^{i}(q_k\cdots q_{k+d-1})^{-1}\gamma_i^d\delta_i^d$, $x=\sum_{i=0}^{m-1}\gamma_i^{md}$, $y=\sum_{i=0}^{m-1}\delta_i^{md}$ and $\epsilon_d=(-1)^{md/2}\prod_{l=1}^{m-1}\prod_{k=1}^{ld}(q_k\cdots q_{k+d-1})^{-1}$. Then $w^m=\epsilon_d xy$. Moreover, by the formula (2-2), any homogeneous element $z=\sum_{i=0}^{m-1}u_i\gamma_i^{s_0}\delta_i^{t_0}\in Z_{gr}(E(\Gamma_q))\setminus k$ can be written as a scalar multiple of $x^iy^jw^l$ with $j+l>0$ since $t_0\geq 1$. The condition $j+l>0$ implies that $x^i, i\geq 1$, does not belong to $Z_{gr}(E(\Gamma_q))$. With the same argument as in [**19**, Lemma 2.3], the elements x,y,w don't have additional relation except $w^m=\epsilon_d xy$ in $Z_{gr}(E(\Gamma_q))$. So $Z_{gr}(E(\Gamma_q))\cong(k[x,y,w]/\langle w^m-\epsilon_d xy\rangle)_{x^*}$.

Case 2. m is odd and char$k\neq 2$.

(i) If d is odd, writing $t_0=\alpha dm+t, s_0=\beta dm+s$, then any homogeneous element $z\in Z_{gr}(E(\Gamma_q))\setminus k$ can be written as

$$\begin{aligned} z &= u_0\sum_{i=0}^{m-1}\prod_{k=1}^{i}(q_k\cdots q_{k+dl/2-1})^{-1}\gamma_i^{dl/2}\delta_i^{dl/2}(\sum_{i=0}^{m-1}\gamma_i^{2dm})^{\alpha/2}(\sum_{i=0}^{m-1}\delta_i^{2dm})^{\beta/2}, \quad \text{or}\\ z &= u_0\sum_{i=0}^{m-1}\prod_{k=1}^{i}(q_k\cdots q_{k+dl/2-1})^{-1}\gamma_i^{d(l/2+m)}\delta_i^{d(l/2+m)}(\sum_{i=0}^{m-1}\gamma_i^{2dm})^{(\alpha-1)/2}(\sum_{i=0}^{m-1}\delta_i^{2dm})^{(\beta-1)/2}\end{aligned}$$

depending on whether α is even or odd. Set $w=\sum_{i=0}^{m-1}\prod_{k=1}^{i}(-1)^i(q_k\cdots q_{k+2d-1})^{-1}\gamma_i^{2d}\delta_i^{2d}$, $x=\sum_{i=0}^{m-1}\gamma_i^{2md}$, $y=\sum_{i=0}^{m-1}\delta_i^{2md}$ and $\epsilon_d=\prod_{l=1}^{m-1}\prod_{k=1}^{2dl}(q_k\cdots q_{k+2d-1})^{-1}$. Then $w^m=\epsilon_d xy$. Moreover, any homogeneous element $z\in Z_{gr}(E(\Gamma_q))\setminus k$ can be written as a scalar multiple of $x^iy^jw^l$ with $j+l>0$. And there is no additional relation in $Z_{gr}(E(\Gamma_q))$ except $w^m=\epsilon_d xy$.

(ii) If d is even and $d\equiv 0(\text{mod } 4)$, then any homogeneous element $z\in Z_{gr}(E(\Gamma_q))\setminus k$ can be written as

$$z=u_0\sum_{i=0}^{m-1}\prod_{k=1}^{i}(q_k\cdots q_{k+dl/2-1})^{-1}\gamma_i^{dl/2}\delta_i^{dl/2}(\sum_{i=0}^{m-1}\gamma_i^{dm})^{\alpha}(\sum_{i=0}^{m-1}\delta_i^{dm})^{\beta}.$$

Set $w=\sum_{i=0}^{m-1}\prod_{k=1}^{i}(q_k\cdots q_{k+d-1})^{-1}\gamma_i^d\delta_i^d$, $x=\sum_{i=0}^{m-1}\gamma_i^{md}$, $y=\sum_{i=0}^{m-1}\delta_i^{md}$ and $\epsilon_d=\prod_{l=1}^{m-1}\prod_{k=1}^{dl}(q_k\cdots q_{k+d-1})^{-1}$. Then $w^m=\epsilon_d xy$. And we can write homogeneous element $z\in Z_{gr}(E(\Gamma_q))$ which is not in k as a scalar multiple of $x^\alpha y^\beta w^{l/2}$ with $\beta+l/2>0$. In particular, any scalar multiple of x^i does not lie in $Z_{gr}(E(\Gamma_q))$, for $i=1,2,\cdots$. Also, there is no additional relation in $Z_{gr}(E(\Gamma_q))$ except $w^m=\epsilon_d xy$.

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*For the referee's convenience, we leave the complete proof of the proposition in the appendix.

(iii) If d is even and $d \equiv 2 (\mathrm{mod}\ 4)$, then we can write any homogeneous element $z \in Z_{gr}(E(\Gamma_q))$ that is not in k as

$$z = u_0 \sum_{i=0}^{m-1} \prod_{k=1}^{i} (-1)^{il} (q_k \cdots q_{k+dl/2-1})^{-1} \gamma_i^{dl/2} \delta_i^{dl/2} (\sum_{i=0}^{m-1} \gamma_i^{dm})^{\alpha} (\sum_{i=0}^{m-1} \delta_i^{dm})^{\beta}.$$

Set $w = \sum_{i=0}^{m-1} \prod_{k=1}^{i} (-1)^i (q_k \cdots q_{k+d/2-1})^{-1} \gamma_i^{d/2} \delta_i^{d/2}$, $x = \sum_{i=0}^{m-1} \gamma_i^{md}$, $y = \sum_{i=0}^{m-1} \delta_i^{md}$ and $\epsilon_d = \prod_{l=1}^{2m-1} \prod_{k=1}^{dl/2} (q_k \cdots q_{k+d/2-1})^{-1}$, then $w^{2m} = \epsilon_d xy$. And we can write homogeneous element $z \in Z_{gr}(E(\Gamma_q))$ which is not in k as a scalar multiple of $x^{\alpha} y^{\beta} w^l$ with $\beta + l > 0$, which implies that any scalar multiple of x^i does not lie in $Z_{gr}(E(\Gamma_q))$, for $i = 1, 2, \cdots$. Again, there is no additional relation in $Z_{gr}(E(\Gamma_q))$ except $w^{2m} = \epsilon_d xy$. □

By [**9, 28**], we know that $\mathrm{HH}^*(\Gamma_q)/\mathcal{N} \cong Z_{gr}(E(\Gamma_q))/\mathcal{N}_Z$, where $\mathcal{N}_Z$ denotes the ideal of $Z_{gr}(E(\Gamma_q))$ generated by all nilpotent elements. It follows directly from the above two propositions that $\mathcal{N}_Z = 0$. As a result, we have, in fact, characterized the structure of $\mathrm{HH}^*(\Gamma_q)/\mathcal{N}$ and provided more counterexamples to Snashall-Solberg's conjecture by the following theorem.

Theorem 2.4. Let $q = (q_0, q_1, \cdots, q_{m-1}) \in (k^*)^m$, and $\zeta = q_0 q_1 \cdots q_{m-1}$. If ζ is a not root of unity, then $\mathrm{HH}^*(\Gamma_q)/\mathcal{N} \cong k$; If ζ is a root of unity, then $\mathrm{HH}^*(\Gamma_q)/\mathcal{N} \cong Z_{gr}(E(\Gamma_q))$ is not finitely generated as algebra.

Proof. From the proposition 2.3, we know that if ζ is a root of unity, then $\mathrm{HH}^*(\Gamma_q)/\mathcal{N} \cong (k[x,y,w]/\langle w^p - \epsilon xy \rangle)_{x^*}$. Note that $x^i y$ lies in $(k[x,y,w]/\langle w^p - \epsilon xy \rangle)_{x^*}$ but x^i does not, for $i = 1, 2, \cdots$, then $x^i y$ can not be generated by the elements of lower degree in $(k[x,y,w]/\langle w^p - \epsilon xy \rangle)_{x^*}$, and thus $\mathrm{HH}^*(\Gamma_q)/\mathcal{N}$ is not finitely generated algebra when ζ is a root of unity. □

3. The Hochschild cohomology ring of $\Lambda_q^{m,n}$

Throughout this section, we assume $\Lambda_q^{m,n} = k\bar{Q}/I_q$, where $\bar{Q}$ is a torus-like finite quiver which has mn vertices $\{(i,j) \mid i \in \mathbb{Z}_n, j \in \mathbb{Z}_m\}$, and $2mn$ arrows: $\{a_{ij} : (i,j) \to (i, j+1)\} \cup \{b_{ij} : (i,j) \to (i+1, j)\}$ pictured as in Figure 2, and $I_q = \langle a_{ij} a_{i,j+1},\ b_{ij} b_{i+1,j},\ a_{ij} b_{i,j+1} + q_{ij} b_{ij} a_{i+1,j} \mid i \in \mathbb{Z}_n, j \in \mathbb{Z}_m \rangle$, $q_{ij} \in k^*$. Denote by e_{ij} the idempotent of $\Lambda_q^{m,n}$ at the vertex (i,j). Note that $\Lambda_q^{m,n}$ is the $\mathbb{Z}_n \times \mathbb{Z}_m$-Galois covering algebra of the quantized exterior algebra A_q if $q_{ij} = q_{00}$ for $i \in \mathbb{Z}_n, j \in \mathbb{Z}_m$.

For $x \in \{e, a, b\}$, define $x_{ij} < x_{pl}$ if and only if $i < p$ or $i = p$ but $j < l$; and set $e_{i_1 j_1} < a_{i_2 j_2} < b_{i_3 j_3}$. Then the length-left-lexicographic order provides an admissible order for $k\bar{Q}$, and $R = \{a_{ij} a_{i,j+1}, b_{ij} b_{i+1,j}, a_{ij} b_{i,j+1} + q_{ij} b_{ij} a_{i+1,j}\}$ forms a noncommutative quadratic reduced Gröber basis of the ideal $I_q = \langle a_{ij} a_{i,j+1}, b_{ij} b_{i+1,j}, a_{ij} b_{i,j+1} + q_{ij} b_{ij} a_{i+1,j} \rangle$, thus $\Lambda_q^{m,n}$ is a Koszul algebra[**26, 27**].

In this section, we first construct a minimal projective bimodule resolution of $\Lambda_q^{m,n}$, and then determine the structure of Hochschild cohomology ring of $\Lambda_q^{m,n}$ when $m = n$ and $\xi = \prod_{i,j=0}^{n-1} q_{ij}$ is not a root of unity, and thus provide another family of counterexamples to Happel's conjecture. For the convenience of notations, we denote by Λ_q^n the algebra $\Lambda_q^{n,n}$ unless otherwise specified in this section. If $n = 1$, then Λ_q^1 is just the quantized exterior algebra A_q; if $n = 2$, the Hochschild homology and cohomology of Λ_q^2 have been considered in [**30**] and the k-dimension of $\mathrm{HH}^*(\Lambda_q^2)$ is 4 in the case that q is not a root of unity. From now on we assume $n \geq 3$ in this section.

Let

$$g^0 = \{g_{0,i,j}^0 = e_{ij}\};$$
$$g^1 = \{g_{0,i,j}^1 = a_{ij},\ g_{1,i,j}^1 = b_{ij}\};$$
$$g^2 = \{g_{0,i,j}^2 = a_{ij} a_{i,j+1},\ g_{1,i,j}^2 = a_{ij} b_{i,j+1} + q_{ij} b_{ij} a_{i+1,j},\ g_{2,i,j}^2 = b_{ij} b_{i+1,j}\}.$$

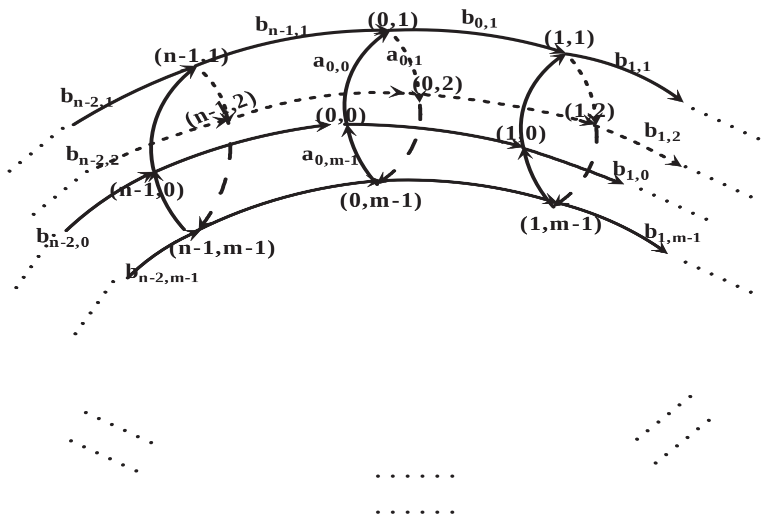

FIGURE 2. The quiver $\bar{Q}$

Moreover, we set $g^l_{-1,i,j} = 0 = g^l_{l+1,i,j}$, and when $l \geq 3$, $g^l = \{g^l_{pij} \mid 0 \leq p \leq l\}$, where

$$g^l_{pij} = a_{ij} g^{l-1}_{p,i,j+1} + q_{ij} q_{i,j+1} \cdots q_{i,j+l-p-1} b_{ij} g^{l-1}_{p-1,i+1,j}. \tag{3-1}$$

Define $P_l = \Lambda^n_q \otimes_E kg^l \otimes_E \Lambda^n_q$, where $\Lambda^n_q = E \oplus \mathrm{r}$ and $E \cong \Lambda^n_q/\mathrm{r} \cong k \times k \times \cdots \times k$. We denote $\otimes_E$ by $\otimes$ for legibility. Set $\tilde{g}^l_{pij} = 1 \otimes g^l_{pij} \otimes 1$ for $0 \leq p \leq l$, $l = 0, 1, 2, \cdots$ and define $d_l : P_l \to P_{l-1}$ for $l \geq 1$ as follows

$$\begin{aligned} d_l(\tilde{g}^l_{pij}) &= a_{ij} \tilde{g}^{l-1}_{p,i,j+1} + q_{ij} q_{i,j+1} \cdots q_{i,j+l-p-1} b_{ij} \tilde{g}^{l-1}_{p-1,i+1,j} + (-1)^l \tilde{g}^{l-1}_{p-1,i,j} b_{i+p-1,j+l-p} \\ &\quad + (-1)^l q_{i,j+l-p-1} \cdots q_{i+p-1,j+l-p-1} \tilde{g}^{l-1}_{p,i,j} a_{i+p,j+l-p-1}. \end{aligned}$$

Lemma 3.1. The complex $(\mathbb{P}, d)$

$$\cdots \to P_{l+1} \xrightarrow{d_{l+1}} P_l \xrightarrow{d_l} \cdots \xrightarrow{d_3} P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \to 0$$

is a minimal projective bimodule resolution of Λ^n_q.

Proof. Let $X = \mathrm{span}\{a_{ij}, b_{ij} \mid i, j \in \mathbb{Z}_n\}$. Since Λ^n_q is Koszul, it suffices to prove that the set g^l forms a k-basis of $K_l := \cap_{s+t=l-2} X^s R X^t$ by [**31**, Sec.9], where, by abuse of notation, R stands for the k-space spanned by the set $\{a_{ij} a_{i,j+1}, b_{ij} b_{i+1,j}, a_{ij} b_{i,j+1} + q_{ij} b_{ij} a_{i+1,j} \mid i, j \in \mathbb{Z}_n\}$.

We will first show that $g^l \subseteq K_l$ by induction on l. It is clear when $l = 2$. Assume that it holds for $l - 1$. It is not difficult but lengthy to verify that

$$g^l_{pij} = g^{l-1}_{p-1,i,j} b_{i+p-1,j+l-p} + q_{i,j+l-p-1} \cdots q_{i+p-1,j+l-p-1} g^{l-1}_{p,i,j} a_{i+p,j+l-p-1}, \ 0 \leq p \leq l.$$

Then by the formula (3-1) and the above formula, we have $g^l \subseteq X K_{l-1} \cap K_{l-1} X \subseteq K_l$.

On the other hand, each element g^l_{pij} in g^l contains exactly p many b-class arrows, and hence the elements in g^l are linearly independent. Moreover, the Koszul dual of Λ^n_q is just the quadratic dual $k\bar{Q}^{op}/I^{\perp}_q$, where $I^{\perp}_q = \langle (a_{ij} b_{i,j+1})^o - q^{-1}_{ij} (b_{ij} a_{i+1,j})^o \rangle$, so the Betti numbers of a minimal projective bimodule resolution of Λ^n_q are $\{b_l = (l+1)n^2\}$, and thus $\dim K_l = (l+1)n^2$, which coincides with the number of elements in g^l.

The differential d is obtained from [**32**] directly. The proof is completed. $\square$

In order to compute the Hochschild cohomology of Λ^n_q when ξ is not a root of unity, we first recall some notations from [**29**]. We say a path α is *uniform* if there exist $(i, j), (i', j') \in \bar{Q}_0$, such

that $\alpha = e_{ij}\alpha e_{i'j'}$. Two paths α and β are said to be *parallel*, and denoted by $\alpha//\beta$, provided that they have the same source and target. If X and Y are sets of some uniform paths in $\bar{Q}$, then $X//Y := \{(\alpha,\beta) \in X \times Y \mid \alpha//\beta\}$ and we denote by $k(X//Y)$ the k-vector space with the set $X//Y$ as basis.

Applying the functor $\mathrm{Hom}_{(\Lambda_q^n)^e}(-,\Lambda_q^n)$ to the minimal projective bimodule resolution $(\mathbb{P},d)$ of Λ_q^n, we obtain the Hochschild cochain complex $C^*(\mathbb{P})$:

$$0 \to \mathrm{Hom}_{(\Lambda_q^n)^e}(P_0,\Lambda_q^n) \xrightarrow{d^1} \cdots \xrightarrow{d^n} \mathrm{Hom}_{(\Lambda_q^n)^e}(P_n,\Lambda_q^n) \xrightarrow{d^{n+1}} \mathrm{Hom}_{(\Lambda_q^n)^e}(P_{n+1},\Lambda_q^n) \xrightarrow{d^{n+2}} \cdots,$$

where $d^i := \mathrm{Hom}_{(\Lambda_q^n)^e}(d_i,\Lambda_q^n), i = 0,1,2,\cdots$.

Let $\mathcal{B} = \{e_{ij}, a_{ij}, b_{ij}, a_{ij}b_{i,j+1} \mid i,j \in \mathbb{Z}_n\}$ be a k-basis of Λ_q^n. Thanks to the isomorphism in [**29**], that is, $k(g^l//\mathcal{B}) \overset{\phi}{\cong} \mathrm{Hom}_{(\Lambda_q^n)^e}(P_l,\Lambda_q^n)$, where $\phi : (g^l_{pij}, x) \mapsto f_{(g^l_{pij},x)}$, $x \in \mathcal{B}$, and $f_{(g^l_{pij},x)}(1 \otimes g^l_{p'i'j'} \otimes 1) = \delta_{pij,p'i'j'}x$. Here $\delta_{pij,p'i'j'}$ denotes the Kronecker sign, that is, $\delta_{pij,p'i'j'} = 1$ if $(p,i,j) = (p',i',j')$ (i.e. $p = p', i = i', j = j'$) and 0 otherwise. Under the isomorphism the complex $(C^*(\mathbb{P}), d^*)$ changes into

$$(M^\bullet,\delta^\bullet) = 0 \to k(g^0//\mathcal{B}) \xrightarrow{\delta^1} \cdots \xrightarrow{\delta^l} k(g^l//\mathcal{B}) \xrightarrow{\delta^{l+1}} k(g^{l+1}//\mathcal{B}) \xrightarrow{\delta^{l+2}} \cdots,$$

where

$$\begin{aligned} & \delta^l(g^{l-1}_{pij},x) = \phi^{-1}d^l\phi(g^{l-1}_{pij},x) \\ = \; & (g^l_{p,i,j-1}, a_{i,j-1}x) + q_{i-1,j}\cdots q_{i-1,j+l-p-2}(g^l_{p+1,i-1,j}, b_{i-1,j}x) + (-1)^l(g^l_{p+1,i,j}, xb_{i+p,j+l-p-1}) \\ & +(-1)^l q_{i,j+l-p-1}\cdots q_{i+p-1,j+l-p-1}(g^l_{p,i,j}, xa_{i+p,j+l-p-1}). \end{aligned}$$

By definition, we know that $\mathrm{HH}^l(\Lambda_q^n) = \mathrm{Ker}\delta^{l+1}/\mathrm{Im}\delta^l$, thus

$$\begin{aligned} \dim_k \mathrm{HH}^l(\Lambda_q^n) &= \dim_k \mathrm{Ker}\delta^{l+1} - \dim_k \mathrm{Im}\delta^l \\ &= \dim_k M^l - \dim_k \mathrm{Im}\delta^{l+1} - \dim_k \mathrm{Im}\delta^l. \end{aligned}$$

Since the set $\mathcal{B} = \{e_{ij}, a_{ij}, b_{ij}, a_{ij}b_{i,j+1}\}$ is a k-basis of Λ_q^n, the elements in $(g^l//\mathcal{B})$ has the form of (g^l_{pij}, x) with $x \in \mathcal{B}$. Note that l stands for the length of g^l_{pij} and p describes the number of b-class arrows appearing in each monomial of g^l_{pij}, and g^l_{pij} is uniform with the source (i,j) and the target $(i+p, j+l-p)$. Thus $(g^l_{pi'j'}//e_{ij})$ implies $(i',j') = (i,j)$ and $l = l_0n$, $p = un$ for some integers u, l_0 with $0 \le u \le l_0$. Similarly, the elements in $g^\bullet$ parallel to $a_{ij}, b_{ij}, a_{ij}b_{i,j+1}$ have the form of $g^{l_0n+1}_{un,i,j}, g^{l_0n+1}_{un+1,i,j}, g^{l_0n+2}_{un+1,i,j}$ respectively, where $0 \le u \le l_0$. Therefore,

$$\dim_k M^l = \dim_k(\Gamma_l//\mathcal{B}) = \begin{cases} (l_0+1)n^2, & \text{if } l = l_0n; \\ 2(l_0+1)n^2, & \text{if } l = l_0n+1; \\ (l_0+1)n^2, & \text{if } l = l_0n+2; \\ 0, & \text{otherwise.} \end{cases} \tag{3-2}$$

Since $M^{l_0n+2} = k(g^{l_0n+2}//\mathcal{B}) = k\{(g^{l_0n+2}_{un+1,i,j}, a_{ij}b_{i,j+1}) \mid 0 \le u \le l_0, i,j \in \mathbb{Z}_n\}$, we have $\delta^{l_0n+3} = 0$ by the definition of $\delta^\bullet$. Also, for $3 < i \le n$, $\delta^{l_0n+i} = 0$ since $M^{l_0n+i-1} = 0$. Thus the complex $(M^\bullet,\delta^\bullet)$ has the forms of

$$0 \to M^0 \xrightarrow{\delta^1} M^1 \xrightarrow{\delta^2} M^2 \xrightarrow{0} \cdots \xrightarrow{0} M^{l_0n} \xrightarrow{\delta^{l_0n+1}} M^{l_0n+1} \xrightarrow{\delta^{l_0n+2}} M^{l_0n+2} \xrightarrow{0} \cdots,$$

where $M^l = k(g^l//\mathcal{B})$. So it suffices to consider $\dim_k \mathrm{Im}\delta^{l_0n+1}$ and $\dim_k \mathrm{Im}\delta^{l_0n+2}$.

The order $<$ on $\mathcal{B}$ induces an order on $(g^l//\mathcal{B})$ as follows: $(g^l_{pij}, x) \prec (g^l_{p'i'j'}, x')$ if and only if $p < p'$, or $p = p'$ but $x < x'$. By abuse of notation, we denote still by δ^l the matrix of the differential δ^l under the ordered bases above. Then by the description of δ^l, δ^{l_0n+1} and δ^{l_0n+2} have

the following form respectively:

$$\delta^{l_0n+1}=\begin{pmatrix} A_0 &&&\\ B_0 &&&\\ & A_1 &&\\ & B_1 &&\\ && \ddots &\\ && \ddots &\\ &&& A_{l_0}\\ &&& B_{l_0}\end{pmatrix};\qquad \delta^{l_0n+2}=\begin{pmatrix} C_0 & D_0 &&&&&\\ && C_1 & D_1 &&&\\ &&&& \ddots & \ddots &\\ &&&&&& C_{l_0} & D_{l_0}\end{pmatrix},$$

where $A_i=\mathrm{diag}\{A_{i0},A_{i1},\cdots A_{i,n-1}\}$, and if we set $r_i=\prod_{j=0}^{n-1}q_{ij}$, $c_j=\prod_{i=0}^{n-1}q_{ij}$, then

$$A_{ij}=\begin{pmatrix} (-1)^lc_0^i & 1 &&\\ & (-1)^lc_1^i & \ddots &\\ && \ddots & 1\\ 1 &&& (-1)^lc_{n-1}^i\end{pmatrix}_{n\times n};$$

$$B_i=\begin{pmatrix} (-1)^lI_n & r_0^{l_0-i}I_n &&\\ & (-1)^lI_n & \ddots &\\ && \ddots & r_{n-2}^{l_0-i}I_n\\ r_{n-1}^{l_0-i}I_n &&& (-1)^lI_n\end{pmatrix}_{n^2\times n^2};$$

$$C_i=\begin{pmatrix} (-1)^lI_n & -r_0^{l_0-i}I_n &&\\ & (-1)^lI_n & \ddots &\\ && \ddots & -r_{n-2}^{l_0-i}I_n\\ -r_{n-1}^{l_0-i}I_n &&& (-1)^lI_n\end{pmatrix}_{n^2\times n^2},$$

where I_n denotes the identity matrix of size $n\times n$; $D_i=\mathrm{diag}\{D_{i0},D_{i1},\cdots,D_{i,n-1}\}$, and

$$D_{ij}=\begin{pmatrix} (-1)^{l+1}c_0^i & 1 &&\\ & (-1)^{l+1}c_1^i & \ddots &\\ && \ddots & 1\\ 1 &&& (-1)^{l+1}c_{n-1}^i\end{pmatrix}_{n\times n}.$$

Lemma 3.2. Suppose that $n\geq 3$ and ξ is not a root of unity. Then

$$\dim_k\mathrm{Im}\delta^{l_0n+1}=\dim_k\mathrm{Im}\delta^{l_0n+2}=\begin{cases} n^2-1, & \text{if } l_0=0;\\ (l_0+1)n^2, & \text{otherwise.}\end{cases}$$

Proof. We first consider the matrix δ^{l_0n+1}. Since, for $0<i\leq l_0$, $\det(A_i)=(\det(A_{i0}))^n=((-1)^{nl}\xi^i+(-1)^{n+1})^n\neq 0$ by the condition that ξ is not a root of unity, the last l_0n^2 columns of δ^{l_0n+1} are linearly independent. We assert that

$$\mathrm{rank}\begin{pmatrix}\dfrac{B_0}{A_0}\end{pmatrix}=\begin{cases} n^2, & \text{if } l_0>0;\\ n^2-1, & \text{if } l_0=0.\end{cases}$$

Indeed, by adding $(-1)^{l+1}r_i^{l_0}$-multiple of the $(i+1)$-th block-column of B_0 to the $(i+2)$-th block-column of B_0 in turn for $i=0,1,\ldots,n-2$, we obtain

$$B_0 \longrightarrow \begin{pmatrix} (-1)^l I_n & & & \\ & \ddots & & \\ & & (-1)^l I_n & \\ r_{n-1}^{l_0} I_n & \cdots & (-1)^{(l+1)(n-2)}(r_{n-1}r_1\cdots r_{n-3})^{l_0} I_n & (-1)^{(l+1)(n-1)}\xi^{l_0} I_n + (-1)^l I_n \end{pmatrix}.$$

Thus $\det(B_0) = \big((-1)^{(l+1)(n-1)}\xi^{l_0} + (-1)^l\big)^n$, which is nonzero if $l_0>0$. Thus $\operatorname{rank}\left(\dfrac{B_0}{A_0}\right) = n^2$ in the case when $l_0>0$. If $l_0=0$, by adding the $(i+1)$-th block-column to the $(i+2)$-th block-column in turn for $i=0,1,\ldots,n-2$, we obtain

$$\left(\frac{B_0}{A_0}\right) \longrightarrow \left(\begin{array}{cccc} -I_n & & & \\ & \ddots & & \\ & & -I_n & \\ I_n & \cdots & I_n & 0 \\ \hline A_{00} & & & A_{00} \\ & \ddots & & \vdots \\ & & A_{0,n-2} & A_{0,n-2} \\ & & & A_{0,n-1} \end{array}\right).$$

Since $A_{00} = A_{01} = \cdots = A_{0,n-1}$ and $\operatorname{rank} A_{00} = n-1$, we have $\operatorname{rank}\delta^{l_0 n+1} = \operatorname{rank}\delta^1 = n^2-1$ in the case when $l_0=0$ as desired. Therefore,

$$\dim_k \operatorname{Im}\delta^{l_0 n+1} = \operatorname{rank}\delta^{l_0 n+1} = l_0 n^2 + \operatorname{rank}\left(\frac{B_0}{A_0}\right) = \begin{cases} n^2-1, & \text{if } l_0=0; \\ (l_0+1)n^2, & \text{otherwise.} \end{cases}$$

We complete the proof of the first part of this lemma.

Next, we consider the rank of $\delta^{l_0 n+2}$. With a similar argument as for $\delta^{l_0 n+1}$, $\det(D_i) = (\det(D_{i0}))^n = ((-1)^{n(l+1)}\xi^i + (-1)^{n+1})^n \neq 0$ for $0 < i \leq l_0$ since ξ is not a root of unity. Therefore, the last $l_0 n^2$ rows of $\delta^{l_0 n+2}$ are linearly independent and it suffices to consider the rank of $(D_0|C_0)$. We claim that

$$\operatorname{rank}\left(\begin{array}{c|c} D_0 & C_0 \end{array}\right) = \begin{cases} n^2, & \text{if } l_0>0; \\ n^2-1, & \text{if } l_0=0. \end{cases}$$

In fact, by adding $(-1)^l r_i^{l_0}$-multiple of the $(i+2)$-th block-row of C_0 to the $(i+1)$-th block-row of C_0 in turn for $i=n-2,n-1,\ldots,0$, we obtain

$$C_0 \longrightarrow \begin{pmatrix} -(-1)^{l(n-1)}\xi^{l_0} I_n + (-1)^l I_n & & & \\ -(-1)^{l(n-2)}(r_{n-1}\cdots r_1)^{l_0} I_n & (-1)^l I_n & & \\ \vdots & & \ddots & \\ -r_{n-1}^{l_0} I_n & & & (-1)^l I_n \end{pmatrix}.$$

So $\det(C_0) = \big(-(-1)^{l(n-1)}\xi^{l_0} + (-1)^l\big)^n \neq 0$, if $l_0>0$. Thus $\operatorname{rank}(D_0|C_0) = n^2$ in the case when $l_0>0$. If $l_0=0$, then, by adding the $(i+1)$-th block-row to the $(i+2)$-th block-row in turn for $i=0,1,\ldots,n-2$, we obtain

$$(D_0|C_0) \longrightarrow \left(\begin{array}{cccc|cccc} D_{00} & & & & I_n & & & \\ & \ddots & & & \vdots & \ddots & & \\ & & D_{0,n-2} & & I_n & & I_n & \\ D_{00} & \cdots & D_{0,n-2} & D_{0,n-1} & 0 & \cdots & 0 & 0 \end{array}\right).$$

Since $D_{00} = D_{01} = \cdots = D_{0,n-1}$ and $\operatorname{rank} D_{00} = n-1$, we have $\operatorname{rank}\delta^{l_0 n+2} = \operatorname{rank}\delta^2 = n^2-1$ in the case when $l_0 = 0$, which proves our claim. Therefore,

$$\dim_k \mathrm{Im}\delta^{l_0 n+2} = \operatorname{rank}\delta^{l_0 n+2} = l_0 n^2 + \operatorname{rank}(D_0|C_0) = \begin{cases} n^2-1, & \text{if } l_0 = 0; \\ (l_0+1)n^2, & \text{otherwise.} \end{cases}$$

$\square$

With the help of Lemma 3.2, we immediately have the following theorem.

Theorem 3.3. If $n \geq 3$ and ξ is not a root of unity, then we have

$$\dim_k \mathrm{HH}^l(\Lambda_q^n) = \begin{cases} 1, & \text{if } l = 0 \text{ or } 2; \\ 2, & \text{if } l = 1; \\ 0, & \text{otherwise.} \end{cases}$$

Thus $\mathrm{HH}^*(\Lambda_q^n)$ is a finite dimensional algebra of dimension 4.

Proof. It follows directly from Lemma 3.2 and the formula

$$\dim_k \mathrm{HH}^l(\Lambda_q^n) = \dim_k M^l - \dim_k \mathrm{Im}\delta^{l+1} - \dim_k \mathrm{Im}\delta^l.$$

$\square$

Remark. Note that our result still holds true for $n = 2$ (cf. [**30**]). Moreover, it also shows that, when ξ is not a root of unity, Λ_q^n provides a family of counterexamples to Happel's question as expected.

Corollary 3.4. If ξ is not a root of unity, then $\mathrm{HH}^*(\Lambda_q^n) \cong \wedge(u,v)$, the exterior algebra.

Proof. For legibility, we do not distinguish the parallel path in M^l with its image in $\mathrm{HH}^l(\Lambda_q^n)$. Moreover, it is straightforward to calculate that $\mathrm{HH}^0(\Lambda_q^n) = \operatorname{span}\{\sum_{i,j}(g^0_{0ij}, e_{ij})\} \cong k$, $\mathrm{HH}^1(\Lambda_q^n) = \operatorname{span}\{\sum_{i,j}(g^1_{0ij}, a_{ij}), \sum_{i,j}(g^1_{1ij}, b_{ij})\}$, $\mathrm{HH}^2(\Lambda_q^n) = \operatorname{span}\{\sum_{i,j}(g^2_{1ij}, a_{ij}b_{i,j+1})\}$. Under the isomorphism $\phi : k(g^l//\mathcal{B}) \to \mathrm{Hom}_{(\Lambda_q^n)^e}(P_l, \Lambda_q^n)$, we have $f^1_a = \sum_{i,j} f_{(g^1_{0ij}, a_{ij})}$ and $f^1_b = \sum_{i,j} f_{(g^1_{1ij}, b_{ij})}$ also form a k-basis of $\mathrm{HH}^1(\Lambda_q^n)$, and $f^2_{ab} = \sum_{i,j} f_{(g^2_{1ij}, a_{ij}b_{i,j+1})}$ a k-basis of $\mathrm{HH}^2(\Lambda_q^n)$. We define bimodule maps

$$\psi_0 : P_1 \to P_0, \quad \begin{cases} \tilde{g}^1_{0,i,j} \mapsto 0, \\ \tilde{g}^1_{1,i,j} \mapsto b_{ij}\tilde{g}^0_{0,i+1,j}; \end{cases}$$

$$\psi_1 : P_2 \to P_1, \quad \begin{cases} \tilde{g}^2_{0,i,j} \mapsto 0, \\ \tilde{g}^2_{1,i,j} \mapsto -q_{ij}b_{ij}\tilde{g}^1_{0,i+1,j}; \\ \tilde{g}^2_{2,i,j} \mapsto -b_{ij}\tilde{g}^1_{1,i+1,j} \end{cases}$$

Now it is easy to check that the following diagram is commutative:

$$\begin{tikzcd}
P_2 \arrow[r, "d_2"] \arrow[d, "\psi_1"'] & P_1 \arrow[d, "\psi_0"'] \arrow[dr, "f^1_b"] & \\
P_1 \arrow[r, "d_1"] \arrow[dr, "f^1_a"'] & P_0 \arrow[r, "\mu"] & \Lambda_q^n \\
& \Lambda_q^n &
\end{tikzcd}$$

where μ is the multiplication. Thus the composition $f^1_a\psi_1 : P_2 \to \Lambda_q^n$ is just the Yoneda product $f^1_a * f^1_b$ in $\mathrm{HH}^2(\Lambda_q^n)$, which is f^2_{ab} and thus is nonzero in $\mathrm{HH}^2(\Lambda_q^n)$. By the graded commutativity of $\mathrm{HH}^*(\Lambda_q^n)$, we have $f^1_a * f^1_b = -f^1_b * f^1_a$, and $f^1_a * f^1_a = 0 = f^1_b * f^1_b$ when $\operatorname{char} k \neq 2$. These still hold by a direct calculation when $\operatorname{char} k = 2$. Denote $u = f^1_a$, $v = f^1_b$ for simplicity. So $\mathrm{HH}^*(\Lambda_q^n) \cong \wedge(u,v)$.

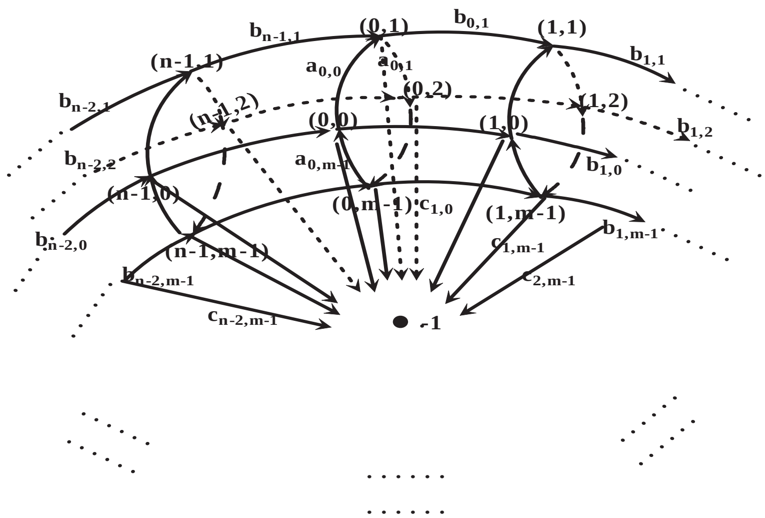

FIGURE 3. The quiver $\tilde{Q}$

4. The graded center of $E(\Gamma_q^{m,n})$

Let $\Gamma_q^{m,n} = k\tilde{Q}/\tilde{I}_q$, where $\tilde{Q}$ is a wheel-like finite quiver with $mn+1$ vertices: $\{(i,j) \mid i \in \mathbb{Z}_n, j \in \mathbb{Z}_m\} \cup \{-1\}$, and $3mn$ arrows: $\{a_{ij} : (i,j) \to (i,j+1)\} \cup \{b_{ij} : (i,j) \to (i+1,j)\} \cup \{c_{ij} : (i,j) \to -1\}$(see Figure 3), and $\tilde{I}_q = \langle a_{ij}a_{i,j+1},\ b_{ij}b_{i+1,j},\ a_{ij}c_{i,j+1},\ a_{ij}b_{i,j+1} + q_{ij}b_{ij}a_{i+1,j}\rangle$, $q_{ij} \in k^*$. In fact, the algebra $\Gamma_q^{m,n}$ can be regarded as a one-point coextension of the algebra $\Lambda_q^{m,n}$ defined in the previous section. Throughout this section, we assume that $\eta = \prod_{i=0}^{n-1}\prod_{j=0}^{m-1} q_{ij}$, and denote by e_{ij} the idempotent of $\Lambda_q^{m,n}$ at the vertex (i,j) and by e_{-1} the idempotent at the vertex -1. In this section, we will describe the graded center of $E(\Gamma_q^{m,n})$ by applying Snashall and Taillefer's method in [**19**] to the algebra $\Gamma_q^{m,n}$.

In a similar way to the previous sections, we can show that $\Gamma_q^{m,n}$ is a Koszul algebra. Moreover, its Koszul dual $E(\Gamma_q^{m,n}) = k\tilde{Q}^{op}/\tilde{I}_q^{\perp}$, where $\tilde{I}_q^{\perp} = \langle (b_{ij}c_{i+1,j})^o,\ (a_{ij}b_{i,j+1})^o - q_{ij}^{-1}(b_{ij}a_{i+1,j})^o\rangle$ and x^o denotes the arrow in $\tilde{Q}^{op}$ corresponding to x in $\tilde{Q}$. Moreover, $E(\Gamma_q^{m,n})$ can be viewed as a quotient of $k\tilde{Q}$ modulo the ideal generated by $b_{ij}c_{i+1,j}, a_{ij}b_{i,j+1} - q_{ij}^{-1}b_{ij}a_{i+1,j}$ for $i \in \mathbb{Z}_n, j \in \mathbb{Z}_m$. Denote still by $Z_{gr}(E(\Gamma_q^{m,n}))$ the graded center of $E(\Gamma_q^{m,n})$.

In this section, we do not differentiate the path in $k\tilde{Q}$ and its image in $E(\Gamma_q^{m,n})$. Since $e_{ij}z = ze_{ij}$ and $e_{-1}z = ze_{-1}$ for any $z \in Z_{gr}(E(\Gamma_q^{m,n}))$, we can write $z = \sum\limits_{i=0}^{n-1}\sum\limits_{j=0}^{m-1} e_{ij}ze_{ij} + e_{-1}ze_{-1}$. Let α_{ij} and β_{ij} denote the path $a_{ij}a_{i,j+1}\cdots a_{i,j+m-1}$ and $b_{ij}b_{i+1,j}\cdots b_{i+n-1,j}$ respectively. Using the relation $a_{ij}b_{i,j+1} = q_{ij}^{-1}b_{ij}a_{i+1,j}$ for $i \in \mathbb{Z}_n, j \in \mathbb{Z}_m$ repeatedly, an element z satisfying $z = e_{ij}ze_{ij}$ can be written as the form $z = u_{ij}\alpha_{ij}^{s_{ij}}\beta_{ij}^{t_{ij}}$ for some $u_{ij} \in k$. Moreover, $e_{-1}ze_{-1} = e_{-1}$.

Noting that $Z_{gr}(E(\Gamma_q^{m,n}))$ can be generated by some elements which are length homogeneous, we denote by $|z|$ the length of such an element z and z must satisfy the following additional conditions:

(1) $a_{ij}z = (-1)^{|z|}za_{ij}$, for $i \in \mathbb{Z}_n, j \in \mathbb{Z}_m$;
(2) $b_{ij}z = (-1)^{|z|}zb_{ij}$, for any $i \in \mathbb{Z}_n, j \in \mathbb{Z}_m$;
(3) $c_{ij}z = (-1)^{|z|}zc_{ij}$, for any $i \in \mathbb{Z}_n, j \in \mathbb{Z}_m$.

Lemma 4.1. For any homogeneous element $z \in Z_{gr}(E(\Gamma_q^{m,n}))$, we have $z \in k$ or z can be written as

$$z = \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} u_{ij}\alpha_{ij}^{s_0}\beta_{ij}^{t_0}$$

with $u_{ij} = (-1)^{(i+j)(ms_0+nt_0)}(\prod_{l=0}^{j-1}\prod_{p=0}^{n-1} q_{pl}^{t_0})(\prod_{p=0}^{i-1}\prod_{l=0}^{m-1} q_{pl}^{-s_0})u_{00} \in k^*$ and $t_0 \geq 1$. Moreover,

$$\eta^{t_0} = (-1)^{m(ms_0+nt_0)}, \eta^{s_0} = (-1)^{n(ms_0+nt_0)}.$$

Proof. We consider the condition (1). If $|z| = 0$, then $z = \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} u_{ij}e_{ij} + u_{-1}e_{-1}$ with $u_{ij}, u_{-1} \in k$, and $a_{ij}z = (-1)^{|z|}za_{ij}$ implies that $u_{ij} = u_{i,j+1}$ for $i \in \mathbb{Z}_n, j \in \mathbb{Z}_m$. If $|z| \neq 0$, then z has the form $z = \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} u_{ij}\alpha_{ij}^{s_{ij}}\beta_{ij}^{t_{ij}}$ with $ms_{ij} + nt_{ij} = ms_{00} + nt_{00}$. Moreover,

$$a_{ij}z = a_{ij}\cdot u_{i,j+1}\alpha_{i,j+1}^{s_{i,j+1}}\beta_{i,j+1}^{t_{i,j+1}} = (q_{ij}\cdots q_{i+n-1,j})^{-t_{i,j+1}}u_{i,j+1}\alpha_{ij}^{s_{i,j+1}}\beta_{ij}^{t_{i,j+1}}a_{ij},$$

and $za_{ij} = u_{ij}\alpha_{ij}^{s_{ij}}\beta_{ij}^{t_{ij}}a_{ij}$. Thus the equality $a_{ij}z = (-1)^{|z|}za_{ij}$ implies that $s_{ij} = s_{i,j+1}$, $t_{ij} = t_{i,j+1}$ and $u_{i,j+1} = (-1)^{ms_{00}+nt_{00}}(q_{ij}\cdots q_{i+n-1,j})^{t_{i,j+1}}u_{ij}$. Recursively, we have $u_{0,0} = u_{0,m} = (-1)^{m(ms_{00}+nt_{00})}\eta^{t_{00}}u_{00}$.

Similarly, the condition (2) implies that if $|z| = 0$, then $u_{i+1,j} = u_{ij}$. Thus we have $z = u_{00}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1} e_{ij} + u_{-1}e_{-1}$ with $u_{00}, u_{-1} \in k$. Moreover, if $|z| \neq 0$, then $s_{i+1,j} = s_{ij}$, $t_{i+1,j} = t_{ij}$ and $(-1)^{ms_{00}+nt_{00}}(q_{ij}\cdots q_{i,j+m-1})^{-s_{ij}}u_{ij} = u_{i+1,j}$. Moreover, we have that $u_{0,0} = u_{n,0} = (-1)^{n(ms_{00}+nt_{00})}\eta^{-s_{00}}u_{00}$ recursively. For legibility of notations, we denote s_{00} and t_{00} by s_0 and t_0 respectively. So, taking the condition (1) into consideration, we have $s_{ij} = s_0$, $t_{ij} = t_0$, and $z = \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} u_{ij}\alpha_{ij}^{s_0}\beta_{ij}^{t_0}$ with $u_{ij} = (-1)^{(i+j)(ms_0+nt_0)}(\prod_{l=0}^{j-1}\prod_{p=0}^{n-1} q_{pl}^{t_0})(\prod_{p=0}^{i-1}\prod_{l=0}^{m-1} q_{pl}^{-s_0})u_{00}$. Moreover, Since $u_{00} \neq 0$, we have $\eta^{t_0} = (-1)^{m(ms_0+nt_0)}$ and $\eta^{s_0} = (-1)^{n(ms_0+nt_0)}$.

Finally, we consider the condition (3). If $|z| = 0$, then $u_{-1}c_{ij} = c_{ij}z = zc_{ij} = u_{00}c_{ij}$, which yields $u_{00} = u_{-1}$, and thus $z = u_{00} \in k$. If $|z| \neq 0$, then $z = \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} u_{ij}\alpha_{ij}^{s_0}\beta_{ij}^{t_0}$. Thus $0 = c_{ij}z = (-1)^{|z|}zc_{ij}$ forces $t_0 \geq 1$ as desired because $\beta_{ij}c_{ij}$ lie in $\tilde{I}_q^{\perp}$ but $\alpha_{ij}c_{ij}$ do not for $i \in \mathbb{Z}_n, j \in \mathbb{Z}_m$. The proof of this lemma is finished. $\square$

With a similar argument as in the proof of Proposition 2.2, if $z \notin k$, then $\eta^{t_0} = (-1)^{m(ms_0+nt_0)}$ and $\eta^{s_0} = (-1)^{n(ms_0+nt_0)}$, which implies that η is a root of unity. Thus we immediately have

Proposition 4.2. If η is not a root of unity, then $Z_{gr}(E(\Gamma_q^{m,n})) = k$.

Proposition 4.3. Let $\eta = \prod_{i=0}^{n-1}\prod_{j=0}^{m-1} q_{ij}$ be a primitive d-th root of unity. If char$k = 2$ or m, n are even, then $Z_{gr}(E(\Gamma_q^{m,n})) \cong k \oplus k[x,y]y$.

Proof. In the case that char$k = 2$ or m, n are even, we have $\eta^{s_0} = \eta^{t_0} = 1$, and thus $d|s_0, d|t_0$ since η is a primitive d-th root of unity. We assume $s_0 = sd, t_0 = td$ for some integers $s \geq 0$ and $t \geq 1$ by $t_0 = td \geq 1$.

Recall that for any homogeneous element $z \in Z_{gr}(E(\Gamma_q^{m,n}))$, if $z \notin k$, then

$$\begin{aligned} z &= u_{00}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}(-1)^{(i+j)(ms_0+nt_0)}(\prod_{p=0}^{i-1}\prod_{l=0}^{m-1} q_{pl}^{-s_0})(\prod_{l=0}^{j-1}\prod_{p=0}^{n-1} q_{pl}^{t_0})\alpha_{ij}^{s_0}\beta_{ij}^{t_0} \\ &= u_{00}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\big((\prod_{p=0}^{i-1}\prod_{l=0}^{m-1} q_{pl})^{-1}\alpha_{ij}\big)^{s_0}\big((\prod_{l=0}^{j-1}\prod_{p=0}^{n-1} q_{pl})\beta_{ij}\big)^{t_0} \\ &= u_{00}\big(\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}(\prod_{p=0}^{i-1}\prod_{l=0}^{m-1} q_{pl})^{-1}\alpha_{ij}\big)^{s_0}\big(\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}(\prod_{l=0}^{j-1}\prod_{p=0}^{n-1} q_{pl})\beta_{ij}\big)^{t_0} \end{aligned}$$

$$= u_{00}\Big(\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}(\prod_{p=0}^{i-1}\prod_{l=0}^{m-1}q_{pl})^{-1}\alpha_{ij}\Big)^{sd}\Big(\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}(\prod_{l=0}^{j-1}\prod_{p=0}^{n-1}q_{pl})\beta_{ij}\Big)^{td}. \tag{4-1}$$

Set $x = \big(\sum\limits_{i=0}^{n-1}\sum\limits_{j=0}^{m-1}(\prod_{p=0}^{i-1}\prod_{l=0}^{m-1}q_{pl})^{-1}\alpha_{ij}\big)^d$ and $y = \big(\sum\limits_{i=0}^{n-1}\sum\limits_{j=0}^{m-1}(\prod_{l=0}^{j-1}\prod_{p=0}^{n-1}q_{pl})\beta_{ij}\big)^d$. Then we have $yx = \eta^{d^2}xy = xy$ by $\alpha_{ij}\beta_{ij} = \eta\beta_{ij}\alpha_{ij}$, and thus z can be written as a scalar multiple of x^sy^t with $t \geq 1$. In addition, since $\{x^sy^{l-s} \mid 0 \leq s \leq l\}$ is a linearly independent set for any fixed l, there is no additional homogeneous relation in $Z_{gr}(E(\Gamma_q^{m,n}))$, and hence $Z_{gr}(E(\Gamma_q^{m,n})) \cong k \oplus k[x,y]y$. □

Proposition 4.4. Suppose that η is a primitive d-th root of unity, char$k \neq 2$ and m,n have the different parity. Then $Z_{gr}(E(\Gamma_q^{m,n})) \cong k \oplus k[x,y]y$.

Proof. Without loss of generality, we assume that n is even and m is odd. Then, by Lemma 4.1, the equalities $\eta^{t_0} = (-1)^{m(ms_0+nt_0)}$ and $\eta^{s_0} = (-1)^{n(ms_0+nt_0)}$ imply $\eta^{s_0} = 1$ and $\eta^{t_0} = (-1)^{ms_0}$. And thus we can write $s_0 = sd$ for some integer s.

(i) If d is even, then $\eta^{t_0} = (-1)^{ms_0} = (-1)^{msd} = 1$, thus $d|t_0$ as well. With the same argument as that in the proof of Proposition 4.3 we have $Z_{gr}(E(\Gamma_q^{m,n})) \cong k \oplus k[x,y]y$ as desired.

(ii) If d is odd, then $\eta^{2t_0} = 1$, which implies that $d|2t_0$ and thus $d|t_0$. We assume that $t_0 = td$ with $t \geq 1$. Since $1 = \eta^{t_0} = (-1)^{ms_0}$ and m is odd, we have s_0 is even, and $s_0 = sd$ implies that s is even as well. As what we have done in the proof of Proposition 4.3, for any homogeneous element $z \in Z_{gr}(E(\Gamma_q^{m,n})) \setminus k$, we have the equality (4-1). Set $x = \big(\sum\limits_{i=0}^{n-1}\sum\limits_{j=0}^{m-1}(\prod_{p=0}^{i-1}\prod_{l=0}^{m-1}q_{pl})^{-1}\alpha_{ij}\big)^{2d}$ and $y = \big(\sum\limits_{i=0}^{n-1}\sum\limits_{j=0}^{m-1}(\prod_{l=0}^{j-1}\prod_{p=0}^{n-1}q_{pl})\beta_{ij}\big)^d$. Then we have $yx = \eta^{2d^2}xy = xy$ and $z = u_{00}x^{s/2}y^t$ with $u_{00} \in k^*$, $t \geq 1$ and $s/2 = 0, 1, 2, \cdots$. Again, there is no additional homogeneous relation in $Z_{gr}(E(\Gamma_q^{m,n}))$. Therefore, $Z_{gr}(E(\Gamma_q^{m,n})) \cong k \oplus k[x,y]y$. □

Proposition 4.5. Let η be a primitive d-th root of unity. If char$k \neq 2$ and both m and n are odd, then

$$Z_{gr}(E(\Gamma_q^{m,n})) \cong \begin{cases} (k \oplus k[x,y]y)^{ev}, & \text{if } d \text{ is odd;} \\ k \oplus k[x,y]y, & \text{otherwise,} \end{cases}$$

where $(k\oplus k[x,y]y)^{ev}$ denotes the subalgebra of $k\oplus k[x,y]y$ spanned by all even degree homogeneous elements as k-vector space.

Proof. If char$k \neq 2$ and m,n are odd, then $\eta^{s_0} = \eta^{t_0} = (-1)^{s_0+t_0}$, and thus $d|2s_0, d|2t_0$.

(i) In the case that d is odd, we have $d|s_0$ and $d|t_0$. We assume that $s_0 = sd$ and $t_0 = td$ for $s \geq 0$ and $t \geq 1$. Moreover, $1 = \eta^{s_0} = \eta^{t_0} = (-1)^{s_0+t_0} = (-1)^{s+t}$ implies that $s+t$ is even. In a similar way to the proof of Proposition 4.3, set $x = \big(\sum\limits_{i=0}^{n-1}\sum\limits_{j=0}^{m-1}(\prod_{p=0}^{i-1}\prod_{l=0}^{m-1}q_{pl})^{-1}\alpha_{ij}\big)^d$ and $y = \big(\sum\limits_{i=0}^{n-1}\sum\limits_{j=0}^{m-1}(\prod_{l=0}^{j-1}\prod_{p=0}^{n-1}q_{pl})\beta_{ij}\big)^d$. Then we have that $yx = \eta^{d^2}xy = xy$, and that $z \in Z_{gr}(E(\Gamma_q^{m,n}))\setminus k$ can be written as a scalar multiple of x^sy^t with $t \geq 1$ and $s+t$ is even. Moreover, $yx = xy$ is the sole relation in $Z_{gr}(E(\Gamma_q^{m,n}))$. So $Z_{gr}(E(\Gamma_q^{m,n})) \cong (k \oplus k[x,y]y)^{ev}$.

(ii) If d is even, then $(d/2)|s_0$ and $(d/2)|t_0$. We write $s_0 = s(d/2)$ and $t_0 = t(d/2)$ with $s \geq 0$ and $t \geq 1$. By $\eta^{s_0} = \eta^{t_0} = (-1)^{s_0+t_0}$, we have $\eta^{s_0+t_0} = 1$, which implies that $d|s_0+t_0$. Since $s_0+t_0 = d(s+t)/2$, we have $s+t$ is even. Thus $1 = (-1)^{d(s+t)/2} = (-1)^{s_0+t_0} = \eta^{s_0} = \eta^{t_0}$, which yields $d|s_0$ and $d|t_0$. Therefore, the rest of the proof in this case is the same as the proof of the Proposition 4.3 and we omit it. So $Z_{gr}(E(\Gamma_q^{m,n})) \cong k \oplus k[x,y]y$. □

From the above four propositions, we have $\mathcal{N}_Z = 0$, where $\mathcal{N}_Z$ denotes the ideal of $Z_{gr}(E(\Gamma_q^{m,n}))$ generated by all nilpotent elements. By the isomorphism $\mathrm{HH}^*(\Gamma_q)/\mathcal{N} \cong Z_{gr}(E(\Gamma_q))/\mathcal{N}_Z$ in [**9, 28**],

we have $\mathrm{HH}^*(\Gamma_q)/\mathcal{N} \cong Z_{gr}(E(\Gamma_q))$. Therefore, as is shown in the following theorem, $\Gamma_q^{m,n}$ provides more counterexamples to Snashall-Solberg's conjecture.

Theorem 4.6. Let $\Gamma_q^{m,n}$ be the algebras defined in the beginning of this section. Then

$$\mathrm{HH}^*(\Gamma_q^{m,n})/\mathcal{N} \cong \begin{cases} k, & \text{if } \eta \text{ is not a root of unity;} \\ (k \oplus k[x,y]y)^{ev}, & \text{if } \mathrm{char} k \neq 2,\ \eta \text{ is a } d\text{-th primitive root of unity and } d,m,n \text{ is odd;} \\ k \oplus k[x,y]y, & \text{otherwise.} \end{cases}$$

As a consequence, if η is a root of unity, then $\mathrm{HH}^*(\Gamma_q^{m,n})/\mathcal{N}$ is not finitely generated as algebra.

Proof. The first part of this theorem follows directly from Propositions 4.2-4.5, and the proof of the second part is similar to that of Theorem 2.4. □

Remark. Our result is still true when $m = 1$ or $n = 1$. Moreover, if $m = n = 1$, the above result coincides with that of [**16, 17**].

Appendix.

In this appendix we give a complete proof of Proposition 2.3, which is a bit subtle modification of the proofs of the propositions 2.4 and 2.5 in [**19**].

Proof of Proposition 2.3. We divide into two cases to finish the proof.

Case 1. m is even or char$k = 2$. In this case we have $\zeta^{t_0} = \zeta^{s_0} = 1$. Since ζ is a primitive d-th root of unity, $d|s_0$ and $d|t_0$. We recall that $s_0 \equiv t_0 (\mathrm{mod}\ m)$, so $t_0 = rm + s_0$, for some integer r. Moreover, we have $u_1 = (-1)^{s_0}(q_1 \cdots q_{t_0})^{-1}u_0 = (-1)^{t_0}(q_1 \cdots q_{s_0})^{-1}u_0$. If m is even or char$k = 2$, then $(-1)^{s_0} = (-1)^{t_0}$, and thus $q_1 q_2 \cdots q_{s_0} = q_1 q_2 \cdots q_{t_0}$. If $s_0 \geq t_0$, then $q_{t_0+1}q_{t_0+2}\cdots q_{s_0} = 1$; on the other hand, if $t_0 \geq s_0$, then $q_{s_0+1}q_{s_0+2}\cdots q_{t_0} = 1$. So in both cases, we have $\zeta^r = 1$, and thus $d|r$. So we can write $t_0 = dhm + s_0$ for some integer h.

For any $z \in Z_{gr}(E(\Gamma_q))$, if z is not in k, $z = \sum_{i=0}^{m-1} u_i \gamma_i^{s_0}\delta_i^{t_0}$ with $t_0 \geq 1$, and $u_i = (-1)^{is_0}\prod_{k=1}^{i}(q_k \cdots q_{k+t_0-1})^{-1}u_0 = (-1)^{it_0}\prod_{k=1}^{i}(q_k \cdots q_{k+s_0-1})^{-1}u_0$ for $i = 1, 2, \ldots, m-1$.

(i) We first consider the case $s_0 = 0$ and $t_0 \geq 1$. Then $z = \sum_{i=0}^{m-1} u_i \delta_i^{t_0} = \sum_{i=0}^{m-1} u_i \delta_i^{dhm}$ with $u_i = (-1)^{it_0}u_0 = u_0$, and thus

$$z = \sum_{i=0}^{m-1} u_0 \delta_i^{dhm} = u_0 (\sum_{i=0}^{m-1} \delta_i^{dm})^h.$$

(ii) When $s_0, t_0 \geq 1$, without loss of generality, we may assume $s_0 \leq t_0$. Then

$$\begin{aligned} z &= \sum_{i=0}^{m-1} (-1)^{it_0} \prod_{k=1}^{i} (q_k \cdots q_{k+s_0-1})^{-1} u_0 \gamma_i^{s_0} \delta_i^{t_0} \\ &= \sum_{i=0}^{m-1} (-1)^{is_0} \prod_{k=1}^{i} (q_k \cdots q_{k+s_0-1})^{-1} u_0 \gamma_i^{s_0} \delta_i^{s_0} \delta_i^{dhm} \\ &= u_0 \Big(\sum_{i=0}^{m-1} (-1)^{is_0} \prod_{k=1}^{i} (q_k \cdots q_{k+s_0-1})^{-1} \gamma_i^{s_0} \delta_i^{s_0} \Big) \Big(\sum_{i=0}^{m-1} \delta_i^{dm} \Big)^h. \end{aligned}$$

We assume $s_0 = \alpha dm + s$, $0 \le s \le dm-1$. Then $(-1)^{s_0} = (-1)^s$, and $q_k q_{k+1} \cdots q_{k+s_0-1} = \zeta^{\alpha d} q_k q_{k+1} \cdots q_{k+s-1} = q_k q_{k+1} \cdots q_{k+s-1}$. And the above equality changes into

$$\begin{aligned} z &= u_0\Big(\sum_{i=0}^{m-1}(-1)^{is}\prod_{k=1}^{i}(q_k\cdots q_{k+s-1})^{-1}\gamma_i^{\alpha dm+s}\delta_i^{\alpha dm+s}\Big)\Big(\sum_{i=0}^{m-1}\delta_i^{dm}\Big)^h \\ &= u_0\Big(\sum_{i=0}^{m-1}(-1)^{is}\prod_{k=1}^{i}(q_k\cdots q_{k+s-1})^{-1}\gamma_i^{s}(\sum_{i=0}^{m-1}\gamma_i^{\alpha dm}\delta_i^{\alpha dm})\delta_i^{s}\Big)\Big(\sum_{i=0}^{m-1}\delta_i^{dm}\Big)^h \qquad (A\text{-}1) \\ &= u_0\Big(\sum_{i=0}^{m-1}(-1)^{is}\prod_{k=1}^{i}(q_k\cdots q_{k+s-1})^{-1}\gamma_i^{s}\delta_i^{s}\Big)\Big(\sum_{i=0}^{m-1}\gamma_i^{dm}\Big)^{\alpha}\Big(\sum_{i=0}^{m-1}\delta_i^{dm}\Big)^{\alpha+h}. \end{aligned}$$

Since $d|s_0$ and $s_0 = \alpha dm + s$, we have $d|s$ and $0 \le s \le dm-1$, and thus $s \in \{0, d, 2d, \cdots, d(m-1)\}$. We assume $s = jd$, and define

$$z_j = \sum_{i=0}^{m-1}(-1)^{ijd}\prod_{k=1}^{i}(q_k\cdots q_{k+jd-1})^{-1}\gamma_i^{jd}\delta_i^{jd}$$

for $0 \le j \le m$. In particular, $z_0 = 1$. Moreover, we have that

$$\begin{aligned} z_j z_1 &= \Big(\sum_{i=0}^{m-1}(-1)^{ijd}\prod_{k=1}^{i}(q_k\cdots q_{k+jd-1})^{-1}\gamma_i^{jd}\delta_i^{jd}\Big)\Big(\sum_{i=0}^{m-1}(-1)^{id}\prod_{k=1}^{i}(q_k\cdots q_{k+d-1})^{-1}\gamma_i^{d}\delta_i^{d}\Big) \\ &= \sum_{i=0}^{m-1}(-1)^{i(j+1)d}\prod_{k=1}^{i}(q_k\cdots q_{k+jd-1})^{-1}\prod_{k=1}^{i}(q_k\cdots q_{k+d-1})^{-1}\gamma_i^{jd}\delta_i^{jd}\gamma_i^{d}\delta_i^{d} \\ &= (-1)^{jd}\prod_{k=1}^{jd}(q_k\cdots q_{k+d-1})^{-1}\Big(\sum_{i=0}^{m-1}(-1)^{i(j+1)d}\prod_{k=1}^{i}(q_k\cdots q_{k+(j+1)d-1})^{-1}\gamma_i^{(j+1)d}\delta_i^{(j+1)d}\Big) \\ &= (-1)^{jd}\prod_{k=1}^{jd}(q_k\cdots q_{k+d-1})^{-1}z_{j+1}. \end{aligned}$$

Thus we have $z_1^j = (-1)^{\sum_{i=1}^{j-1} id}(\prod_{l=1}^{j-1}\prod_{k=1}^{ld}(q_k\cdots q_{k+d-1})^{-1})z_j$, for $j = 1, 2, \cdots, m$. In particular,

$$\begin{aligned} z_1^m &= (-1)^{\sum_{i=1}^{m-1} id}\Big(\prod_{l=1}^{m-1}\prod_{k=1}^{ld}(q_k\cdots q_{k+d-1})^{-1}\Big)z_m \\ &= (-1)^{md/2}\Big(\prod_{l=1}^{m-1}\prod_{k=1}^{ld}(q_k\cdots q_{k+d-1})^{-1}\Big)\Big(\sum_{i=0}^{m-1}(-1)^{imd}\prod_{k=1}^{i}(q_k\cdots q_{k+md-1})^{-1}\gamma_i^{md}\delta_i^{md}\Big) \\ &= (-1)^{md/2}\Big(\prod_{l=1}^{m-1}\prod_{k=1}^{ld}(q_k\cdots q_{k+d-1})^{-1}\Big)\Big(\sum_{i=0}^{m-1}\gamma_i^{md}\Big)\Big(\sum_{i=0}^{m-1}\delta_i^{md}\Big). \end{aligned}$$

Set $x = \sum_{i=0}^{m-1}\gamma_i^{md}$, $y = \sum_{i=0}^{m-1}\delta_i^{md}$, $w = z_1 = \sum_{i=0}^{m-1}(-1)^{id}\prod_{k=1}^{i}(q_k\cdots q_{k+d-1})^{-1}\gamma_i^d\delta_i^d$ and $\epsilon_d = (-1)^{md/2}\prod_{l=1}^{m-1}\prod_{k=1}^{ld}(q_k\cdots q_{k+d-1})^{-1}$. Then $w^m = \epsilon_d xy$. Moreover, by the formula (A-1), we have $z \in k$ or z has the form $z = u_0' w^j x^\alpha y^{\alpha+h}$for any homogeneous element $z \in Z_{gr}(E(\Gamma_q))$, where $u_0' \in k$, $s_0 = s + \alpha dm = (j+\alpha m)d > 0$, that is, $j + \alpha m > 0$, and thus $j + \alpha > 0$. Similarly, if $s_0 \ge t_0$, then $z = u_0'' w^j x^{\alpha+h} y^\alpha$ wit h$u_0'' \in k$ and $j + \alpha > 0$. Therefore, in both cases, any homogeneous element $z \in Z_{gr}(E(\Gamma_q)) \setminus k$ can be written as a scalar multiple of $x^i y^j w^l$ with $j + l > 0$ and $w^m = \epsilon_d xy$. In particular, any scalar multiple of x^i does not lie in $Z_{gr}(E(\Gamma_q))$, for $i = 1, 2, \cdots$.

As what Snashall and Taillefer have done in [**19**, Lemma 2.3], we claim that the elements x, y, w don't have additional relation except $w^m = \epsilon_d xy$ in $Z_{gr}(E(\Gamma_q))$.

Indeed, since the elements $x^i y^{n-i}$ have different degree, for $i = 0, 1, \cdots, n$, thus they are linearly independent in $Z_{gr}(E(\Gamma_q))$. So any additional relation in $Z_{gr}(E(\Gamma_q))$ is length homogeneous

of the form

$$f_0(x,y)+f_1(x,y)w+\cdots+f_{m-1}w^{m-1}=0, \qquad (A\text{-}2)$$

where $f_i(x,y)=\sum_{j=0}^{n_i}k_{ij}x^jy^{n_i-j}\in k[x,y]$, and $|f_0(x,y)|=|f_1(x,y)|+|w|$, which implies $n_0|y|=n_1|y|+|w|$, and thus $n_0md=n_1md+2d$, that is, $n_0m=n_1m+2$.

If $m=1$, then $w=\epsilon_dxy$, and thus any element $z\in Z_{gr}(E(\Gamma_q))$ can be generated by x,y. So there is no additional relation in $Z_{gr}(E(\Gamma_q))$.

Now we consider the case $m\geq 2$. $n_0m=n_1m+2$ implies $m=2$ and $n_0=n_1+1$. Then $|x|=|y|=|w|=2d$, and we may choose the minimal n_0 such that $f_0(x,y)+f_1(x,y)w=0$ with $|f_0(x,y)|=2n_0d$ and $|f_1(x,y)|=2(n_0-1)d$. Since $x^{n_0}\notin Z_{gr}(E(\Gamma_q))$, $f_0(x,y)=\sum_{j=0}^{n_0-1}k_{0j}x^jy^{n_0-j}$ and $f_1(x,y)=\sum_{i=0}^{n_0-1}k_{1j}x^jy^{n_0-j-1}$. Then $f_0^2(x,y)=f_1^2(x,y)w^2=\epsilon_df_1^2(x,y)xy$. Comparing the coefficients of y^{2n_0} and $x^{2n_0-1}y$, we have $k_{00}=k_{1,n_0-1}=0$, and then $f_1(x,y)=\sum_{j=0}^{n_0-2}k_{1j}x^jy^{n_0-j-1}$ and $f_0(x,y)=\epsilon_d^{-1}f_0^{'}(x,y)w^2$ with $f_0^{'}(x,y)=\sum_{j=0}^{n_0-1}k_{0,j+1}x^jy^{n_0-j-1}$, thus $\epsilon_d^{-1}f_0^{'}(x,y)w+f_1(x,y)=0$, which contradicts to the minimality of n.

Case 2. m is odd and char$k\neq 2$. By the conditions $\zeta^{s_0}=(-1)^{mt_0}$ and $\zeta^{t_0}=(-1)^{ms_0}$, we know that $\zeta^{2s_0}=\zeta^{2t_0}=1$. Since ζ is a primitive d-th root of unity, $d|2s_0$, and $d|2t_0$. Recall that $s_0\equiv t_0 (\text{mod } m)$, that is, $s_0=t_0+rm$ for some integer r, and $u_1=(-1)^{s_0}(q_1\cdots q_{t_0})^{-1}u_0=(-1)^{t_0}(q_1\cdots q_{s_0})^{-1}u_0$. If $s_0\geq t_0$, then $q_{t_0+1}q_{t_0+2}\cdots q_{s_0}=(-1)^{s_0-t_0}$; on the other hand, if $t_0\geq s_0$, then $q_{s_0+1}q_{s_0+2}\cdots q_{t_0}=(-1)^{t_0-s_0}$. So, in both cases, we have $\zeta^{2r}=1$, and thus $d|2r$. Then $dm|2(t_0-s_0)$. We assume that $s_0=\alpha dm+s$ and $t_0=\beta dm+t$, where $0\leq s,t\leq dm-1$, then $dm|2(s-t)$, without loss of generality, we assume $s\geq t$, then $2(s-t)=0$ or $2(s-t)=dm$.

Now we assert that $2(s-t)=0$ and thus $t=s$. Otherwise, we will have $2(s-t)=dm$. Since m is odd and d is even, $s-t$ and s_0-t_0 have the same parity. Moreover, $(-1)^{s_0-t_0}=\zeta^r=\zeta^{(s_0-t_0)/m}=\zeta^{(\alpha-\beta)d+(s-t)/m}=\zeta^{(s-t)/m}=\zeta^{d/2}=-1$. Therefore, $s-t$ is odd and $d/2$ is odd. We can also get the equality $(-1)^{s_0+t_0}=(-1)^{m(s_0+t_0)}=(-1)^{ms_0}(-1)^{mt_0}=\zeta^{s_0+t_0}=\zeta^{s+t}=\zeta^{2t+s-t}=\zeta^{2t+(dm)/2}=\zeta^{2t}(-1)^m=-\zeta^{2t}$. So $\zeta^{4t}=1$, and thus $d|4t$ and $(d/2)|2t$. Moreover, since $d/2$ is odd, $(d/2)|t$. We assume that $t=ld/2$ for some integer l. If t is even, then l is even, and we have $1=(-1)^t=(-1)^{t_0}=\zeta^{t_0}=\zeta^s=\zeta^{t+(s-t)}=\zeta^{(l+m)d/2}=\zeta^{l+m}=-1$, this yields a contradiction. Therefore, t is odd, then l is odd, $s=t+(s-t)=(l+m)d/2$ is even, and we have $1=(-1)^s=(-1)^{s_0}=\zeta^{t_0}=\zeta^t=\zeta^{ld/2}=(-1)^l=-1$, a contradiction again. So $2(s-t)=0$ and thus $t=s$ as desired.

Since $t_0=\alpha dm+t, s_0=\beta dm+s$ and $t=s$, we have $s_0-t_0=(\alpha-\beta)dm$ and $1=\zeta^{(\alpha-\beta)dm}=\zeta^{s_0-t_0}=(-1)^{m(t_0-s_0)}=(-1)^{t_0-s_0}=(-1)^{(\beta-\alpha)dm}$. So αdm and βdm have the same parity, and thus αd and βd have the same parity. By squaring the equality $\zeta^t=\zeta^{t_0}=(-1)^{ms_0}$, we know $\zeta^{2t}=1$, and thus $d|2t$ with $0\leq 2t<2dm$. We assume $2t=dl$ for some integer $0\leq l<2m$.

Now, we will describe any homogeneous element in $Z_{gr}(E(\Gamma_q))$. We recall that if z is not in k, $z=\sum_{i=0}^{m-1}(-1)^{it_0}\prod_{k=1}^{i}(q_k\cdots q_{k+s_0-1})^{-1}u_0\gamma_i^{s_0}\delta_i^{t_0}$ with $t_0\geq 1$.

(i) If d is odd, then by $2t=dl$, we have l is even and since $\alpha d,\beta d$ have the same parity, $1=\zeta^{dl/2}=\zeta^t=\zeta^{t_0}=(-1)^{ms_0}=(-1)^{s_0}=(-1)^{\alpha dm+s}=(-1)^{\alpha d+t}=(-1)^{(\alpha+l/2)d}=(-1)^{(\beta+l/2)d}$. So $(\alpha+l/2)d$ and $(\beta+l/2)d$ are even with $0\leq l<m$. If α is even, then $l/2$ and thus $t_0=\beta dm+dl/2$ is even. So we have

$$\begin{aligned}
z &= u_0\sum_{i=0}^{m-1}\prod_{k=1}^{i}(q_k\cdots q_{k+s_0-1})^{-1}\gamma_i^{s_0}\delta_i^{t_0}\\
&= u_0\sum_{i=0}^{m-1}\prod_{k=1}^{i}(q_k\cdots q_{k+dl/2-1})^{-1}\gamma_i^{\alpha dm+dl/2}\delta_i^{\beta dm+dl/2}\\
&= u_0\sum_{i=0}^{m-1}\prod_{k=1}^{i}(q_k\cdots q_{k+dl/2-1})^{-1}\gamma_i^{dl/2}\delta_i^{dl/2}(\sum_{i=0}^{m-1}\gamma_i^{2dm})^{\alpha/2}(\sum_{i=0}^{m-1}\delta_i^{2dm})^{\beta/2}.
\end{aligned}$$

Similarly, if α is odd, then $l/2$ is odd, and t_0 is even, we have

$$\begin{aligned} z &= u_0 \sum_{i=0}^{m-1} \prod_{k=1}^{i} (q_k \cdots q_{k+s_0-1})^{-1} \gamma_i^{s_0} \delta_i^{t_0} \\ &= u_0 \sum_{i=0}^{m-1} \prod_{k=1}^{i} (q_k \cdots q_{k+dl/2-1})^{-1} \gamma_i^{\alpha dm+dl/2} \delta_i^{\beta dm+dl/2} \end{aligned}$$

$$= \sum_{i=0}^{m-1} \prod_{k=1}^{i} (q_k \cdots q_{k+dl/2-1})^{-1} u_0 \gamma_i^{d(l/2+m)} \delta_i^{d(l/2+m)} (\sum_{i=0}^{m-1} \gamma_i^{2dm})^{(\alpha-1)/2} (\sum_{i=0}^{m-1} \delta_i^{2dm})^{(\beta-1)/2}.$$

As what we have done in the case 1, we define

$$z_j = \sum_{i=0}^{m-1} \prod_{k=1}^{i} (q_k \cdots q_{k+2dj-1})^{-1} \gamma_i^{2dj} \delta_i^{2dj}$$

for $j = 1, 2, \cdots, m$, then $z_1 = \sum_{i=0}^{m-1} \prod_{k=1}^{i} (q_k \cdots q_{k+2d-1})^{-1} \gamma_i^{2d} \delta_i^{2d}$. Moreover, by a straightforward verification,

$$z_1 z_j = \prod_{k=1}^{2dj} (q_k \cdots q_{k+2d-1})^{-1} z_{j+1},$$

for $j = 1, 2, \cdots, m$. Thus, $z_1^j = \prod_{l=1}^{j-1} \prod_{k=1}^{2dl} (q_k \cdots q_{k+2d-1})^{-1} z_j$, for $j = 1, 2, \cdots, m$. In particular,

$$\begin{aligned} z_1^m &= \prod_{l=1}^{m-1} \prod_{k=1}^{2dl} (q_k \cdots q_{k+2d-1})^{-1} z_m \\ &= \prod_{l=1}^{m-1} \prod_{k=1}^{2dl} (q_k \cdots q_{k+2d-1})^{-1} (\sum_{i=0}^{m-1} \gamma_i^{2dm}) (\sum_{i=0}^{m-1} \delta_i^{2dm}). \end{aligned}$$

Set $x = \sum_{i=0}^{m-1} \gamma_i^{2md}$, $y = \sum_{i=0}^{m-1} \delta_i^{2md}$, $w = z_1$ and $\epsilon_d = \prod_{l=1}^{m-1} \prod_{k=1}^{2dl} (q_k \cdots q_{k+2d-1})^{-1}$. Then $w^m = \epsilon_d xy$. Moreover, if α is even, then any $z \in Z_{gr}(E(\Gamma_q)) \setminus k$ is a scalar multiple of $x^{\alpha/2} y^{\beta/2} w^{l/4}$ with $\beta/2 + l/4 > 0$ (because $t_0 = \beta dm + dl/2 > 0$). Similarly, if α is odd and $z \in Z_{gr}(E(\Gamma_q)) \setminus k$, then z is a scalar multiple of $x^{(\alpha-1)/2} y^{(\beta-1)/2} w^{(l/2+m)/2}$ with $(\beta-1)/2 + (l/2+m)/2 > 0$. In both cases, $z \in Z_{gr}(E(\Gamma_q)) \setminus k$ can be written as a scalar multiple of $x^i y^j w^l$ with $j + l > 0$ and $w^m = \epsilon_d xy$. Note that any scalar multiple of x^i does not belong to $Z_{gr}(E(\Gamma_q))$, for $i = 1, 2, \cdots$.

With a similar argument as in the case 1, we can assert that x, y, w have no additional homogeneous relation except $w^m = \epsilon_d xy$. Indeed, it suffices to note that $n_0|x| = 2n_0 dm = |f_0(x, y)| = |f_1(x, y)| + |w| = n_1|y| + |w| = 2n_1 dm + 4d$, and thus $(n_0 - n_1)m = 2$ has no solution in $\mathbb{Z}$. So there is no additional homogeneous relation of the form (A-2) as required.

(ii) Now we consider the case $d \equiv 0 (\text{mod } 4)$. We assert that l is even with $0 \leq l/2 < m$, and thus t_0 is even. Otherwise, if l is odd, then, by $-1 = (-1)^l = (\zeta^{(d/2)})^l = \zeta^t = \zeta^{t_0} = (-1)^{ms_0} = (-1)^{s_0} = (-1)^{\alpha dm+s} = (-1)^s = (-1)^t = (-1)^{dl/2} = (-1)^{d/2}$, we have that $d/2$ is odd, which contradicts to $d \equiv 0 (\text{mod } 4)$. Therefore, for any given homogeneous element $z \in Z_{gr}(E(\Gamma_q)) \setminus k$, we have

$$\begin{aligned} z &= u_0 \sum_{i=0}^{m-1} \prod_{k=1}^{i} (q_k \cdots q_{k+dl/2-1})^{-1} \gamma_i^{\alpha dm+dl/2} \delta_i^{\beta dm+dl/2} \\ &= u_0 \sum_{i=0}^{m-1} \prod_{k=1}^{i} (q_k \cdots q_{k+dl/2-1})^{-1} \gamma_i^{dl/2} \delta_i^{dl/2} (\sum_{i=0}^{m-1} \gamma_i^{dm})^{\alpha} (\sum_{i=0}^{m-1} \delta_i^{dm})^{\beta}. \end{aligned}$$

We define $z_j = \sum_{i=0}^{m-1}\prod_{k=1}^{i}(q_k\cdots q_{k+dj-1})^{-1}\gamma_i^{dj}\delta_i^{dj}$, for $j = 1,2,\cdots,m$. Then it is clear that $z_jz_1 = \prod_{k=1}^{dj}(q_k\cdots q_{k+d-1})^{-1}z_{j+1}$, for $j = 1,2,\cdots,m$. Thus, $z_1^j = \prod_{l=1}^{j-1}\prod_{k=1}^{dl}(q_k\cdots q_{k+d-1})^{-1}z_j$, for $j = 1,2,\cdots,m$.

Set $x = \sum_{i=0}^{m-1}\gamma_i^{md}$, $y = \sum_{i=0}^{m-1}\delta_i^{md}$, $w = z_1$ and $\epsilon_d = \prod_{l=1}^{m-1}\prod_{k=1}^{dl}(q_k\cdots q_{k+d-1})^{-1}$. Then $w^m = \epsilon_d xy$. And we can write any homogeneous element $z \in Z_{gr}(E(\Gamma_q)) \setminus k$ as a scalar multiple of $x^\alpha y^\beta w^{l/2}$ with $\beta + l/2 > 0$. In particular, any scalar multiple of x^i is not in $Z_{gr}(E(\Gamma_q))$, for $i = 1,2,\cdots$.

Similarly, we can also prove that x,y,w have no additional relation except $w^m = \epsilon_d xy$. Thus we have $Z_{gr}(E(\Gamma_q)) \cong (k[x,y,w]/\langle w^m - \epsilon_d xy\rangle)_{x^*}$, where $\epsilon_d = \prod_{l=1}^{m-1}\prod_{k=1}^{ld}(q_k\cdots q_{k+d-1})^{-1}$.

(iii) If d is even with $d \equiv 2(\mathrm{mod}\ 4)$, then $d/2$ is odd, and t_0 and l have the same parity by $t_0 = \beta dm + ld/2$, where $0 \le l < 2m$. So we can write any homogeneous element $z \in Z_{gr}(E(\Gamma_q))$ that is not in k as

$$\begin{aligned} z &= u_0\sum_{i=0}^{m-1}\prod_{k=1}^{i}(-1)^{il}(q_k\cdots q_{k+dl/2-1})^{-1}\gamma_i^{\alpha dm+dl/2}\delta_i^{\beta dm+dl/2} \\ &= u_0\sum_{i=0}^{m-1}\prod_{k=1}^{i}(-1)^{il}(q_k\cdots q_{k+dl/2-1})^{-1}\gamma_i^{dl/2}\delta_i^{dl/2}(\sum_{i=0}^{m-1}\gamma_i^{dm})^\alpha(\sum_{i=0}^{m-1}\delta_i^{dm})^\beta. \end{aligned}$$

Similarly, define

$$z_j = \sum_{i=0}^{m-1}\prod_{k=1}^{i}(-1)^{ij}(q_k\cdots q_{k+dj/2-1})^{-1}\gamma_i^{dj/2}\delta_i^{dj/2},$$

for $j = 1,2,\cdots,2m$. Then we can verify that $z_jz_1 = \prod_{k=1}^{dj/2}(q_k\cdots q_{k+d/2-1})^{-1}z_{j+1}$, and thus $z_1^j = \prod_{l=1}^{j-1}\prod_{k=1}^{dl/2}(q_k\cdots q_{k+d/2-1})^{-1}z_j$, for $j = 1,2,\cdots,2m$.

Set $x = \sum_{i=0}^{m-1}\gamma_i^{md}$, $y = \sum_{i=0}^{m-1}\delta_i^{md}$, $w = z_1$ and $\epsilon_d = \prod_{l=1}^{2m-1}\prod_{k=1}^{dl/2}(q_k\cdots q_{k+d/2-1})^{-1}$. Then $w^{2m} = \epsilon_d xy$. And we can write any homogeneous element $z \in Z_{gr}(E(\Gamma_q)) \setminus k$ as a scalar multiple of $x^\alpha y^\beta w^l$ with $\beta + l > 0$ since $t_0 = \beta dm + ld/2 > 0$.

Again, there is no additional relation in $Z_{gr}(E(\Gamma_q))$ except $w^{2m} = \epsilon_d xy$, and any scalar multiple of x^i is not in $Z_{gr}(E(\Gamma_q))$, for $i = 1,2,\cdots$. So we have $Z_{gr}(E(\Gamma_q)) \cong (k[x,y,w]/\langle w^{2m} - \epsilon_d xy\rangle)_{x^*}$, where $\epsilon_d = \prod_{l=1}^{2m-1}\prod_{k=1}^{ld/2}(q_k\cdots q_{k+d/2-1})^{-1}$ in this case. $\square$

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